CHAPTER-10-VECTOR ALGEBRA

Gist/Summary of the lesson (Definitions and Formulae)

PRODUCT OF TWO VECTORS

There are two types of products between two vectors.

1. Scalar Product (OR) Dot Product.
2. Vector Product (OR) Cross Product.

Scalar Product:

If $\vec{a}$, $\vec{b}$ are two non-zero vectors and if the angle between them is θ then $\vec{a}.\vec{b}$ is defined as $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\ Cos\theta$

Case (1). If $\vec{a}$, $\vec{b}$ are Like vectors then $\theta = 0°$.

$$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\ Cos\ 0° = |\vec{a}||\vec{b}|$$

Case (2). If $\vec{a}$, $\vec{b}$ are Unlike vectors then $\theta = 180°$.

$$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\ Cos180° = -|\vec{a}||\vec{b}|$$

Case(3). If $\vec{a}$, $\vec{b}$ are Perpendicular vectors then $\theta = 90°$.

$$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\ Cos90° = 0 \Rightarrow \vec{a}.\vec{b} = 0$$

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ are two vectors then the Scalar Product $\vec{a}.\vec{b}$ is defined as $\vec{a}.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$ (Scalar)

Properties of Dot Product:

1. $\vec{a}.\vec{b} = \vec{b}.\vec{a}$ 2. $\vec{a}.(-\vec{b}) = -(\vec{a}.\vec{b})$ 3. $\vec{a}.\vec{a} = |\vec{a}||\vec{a}|\ Cos\ 0° = |\vec{a}|^2 = a^2$

4. $(m\vec{a}).\vec{b} = m(\vec{a}.\vec{b}) = \vec{a}.m(\vec{b})$

Angle between vectors:

Let θ be the angle between two non-zero vectors $\vec{a}$, $\vec{b}$. Then

$$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\ Cos\theta \Rightarrow Cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$$

Composition Table of Dot Product:

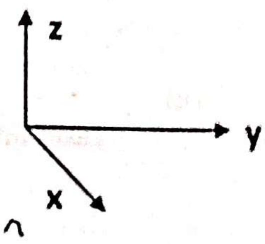
•	$\hat{i}$	$\hat{j}$	$\hat{k}$
$\hat{i}$	1	0	0
$\hat{j}$	0	1	0
$\hat{k}$	0	0	1

Projection of a Vector:

Let $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$. Let θ be the angle between $\vec{a}$ and $\vec{b}$.

∴The Projection of $\vec{b}$ on $\vec{a}$ $= OM = \frac{\vec{a}.\vec{b}}{|\vec{a}|}$, Similarly the Projection of $\vec{a}$ on $\vec{b}$ $= \frac{\vec{a}.\vec{b}}{|\vec{b}|}$

Vector (or Cross) Product of two vectors:



The vector product of two non – zero vectors $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a}\ X\ \vec{b}$ and is defined as

$\vec{a}\ X\ \vec{b} = |\vec{a}||\vec{b}|\ Sin\theta\ \hat{n}$

Where θ is angle between $\vec{a}$ and $\vec{b}$, where $0 \leq \theta \leq \pi$ and $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ such that $\vec{a}$, $\vec{b}$ and $\hat{n}$ form a right handed system.

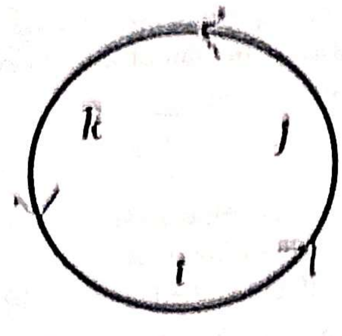
(i). If either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ then $\vec{a} \times \vec{b} = \vec{0}$

(ii). $\vec{a} \times \vec{b}$ is a vector.

(iii). If $\vec{a}$ and $\vec{b}$ are parallel then $\vec{a} \times \vec{b} = \vec{0}$

(iv). If $\theta = 90°$ then $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|$

Composition Table of Cross Product:



X	$\hat{i}$	$\hat{j}$	$\hat{k}$
$\hat{i}$	0	$\hat{k}$	$-\hat{j}$
$\hat{j}$	$-\hat{k}$	0	$\hat{i}$
$\hat{k}$	$\hat{j}$	$-\hat{i}$	0

Angle between vectors:

Let θ be the angle between two non-zero vectors $\vec{a}$ and $\vec{b}$. Then

$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta\, \hat{n} \Rightarrow |\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta \Rightarrow \sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}$

Unit vector ⊥ both the vectors $\vec{a}$ and $\vec{b}$:

$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta\, \hat{n}$ ...(1) then

$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$ .....(2) (1) ÷ (2) $\hat{n} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ are two vectors then the Vector Product $\vec{a} \times \vec{b}$ is defined as $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

Area of a Parallelogram: $|\vec{a} \times \vec{b}|$

Note: Area of a triangle with $\vec{a}$ and $\vec{b}$ as adjacent sides = Area $= \frac{1}{2}|\vec{a} \times \vec{b}|$

Note: When diagonals $\vec{d_1}$ and $\vec{d_2}$ are given, Area of a Parallelogram $= \frac{1}{2}|\vec{d_1} \times \vec{d_2}|$

Relation between Dot and Cross Product:

$$(\vec{a}.\vec{b})^2 + |\vec{a} \times \vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2$$

Note: Stress on Position Vector Concept

Convert a position vector to Cartesian form of a point and vice-versa

Position Vector (Vector Geometry)	Point in 3D Geometry(Cartesian Form)
$\vec{OP} = x\hat{i} + y\hat{j} + z\hat{k}$	P=(x,y,z)
$\vec{OA} = 2\hat{i} + 3\hat{j} - \hat{k}$	A=(2,3,-1)
	C=(1,-2,-3)
$\vec{OX} = 3\hat{i} - 5\hat{j} + 4\hat{k}$	
	Y=(-2,-5,-7)

Activity to remember the concepts:

Requirement	If one vector is given in the Question (Not applicable to position vector) $\vec{r} = a\vec{i} + b\vec{j} + c\vec{k}$	If two points are given in the Question $A(x_1,y_1,z_1)$ and $B(x_2,y_2,z_2)$ **OR** $\vec{OA} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ & $\vec{OB} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$
Direction Ratios of a vector		
Magnitude of a vector		
Direction Cosines of a vector		

Unit Vector in the direction of		
A vector with magnitude k units and in the direction of		

MULTIPLE CHOICE QUESTIONS

1. If the position vectors of the vertices of a triangle be $6i + 4j + 5k$, $4i + 5j + 6k$ and $5i + 6j + 4k$, then the triangle is
(a) Right angled (b) Isosceles (c) Equilateral (d) None of these
Sol: (c) Equilateral, since each side is of length $\sqrt{6}$.

2. The perimeter of the triangle whose vertices have the position vectors $(i + j + k)$, $(5i + 3j - 3k)$ and $(2i + 5j + 9k)$, is given by
(a) $15 + \sqrt{157}$ (b) $15 - \sqrt{157}$ (c) $\sqrt{15} - \sqrt{157}$ (d) $\sqrt{15} + \sqrt{157}$ Sol: (a)
$a = 4i + 2j - 4k \Rightarrow |a| = \sqrt{16 + 16 + 4} = 6$
$b = -3i + 2j + 12k \Rightarrow |b| = \sqrt{144 + 4 + 9} = \sqrt{157}$
$c = -i - 4j - 8k \Rightarrow |c| = \sqrt{64 + 16 + 1} = 9$
Hence perimeter is $15 + \sqrt{157}$.

3. The position vectors of two points A and B are $i + j - k$ and $2i - j + k$ respectively. Then $|\overrightarrow{AB}| =$
(a) 2 (b) 3 (c) 4 (d) 5 Sol: (b)
$\overrightarrow{AB} = i - 2j + 2k \Rightarrow |\overrightarrow{AB}| = 3$.

4. The magnitudes of mutually perpendicular forces a, b and c are 2, 10 and 11 respectively. Then the magnitude of its resultant is
(a) 12 (b) 15 (c) 9 (d) None Sol: (b)
$R = \sqrt{4 + 100 + 121} = 15$.

5. The system of vectors $\hat{i}, \hat{j}, \hat{k}$ is
(a) Orthogonal (b) Coplanar (c) Collinear (d) None of these Sol: (a)
It is a fundamental concept.

6. The direction cosines of the resultant of the vectors $(i + j + k)$, $(-i + j + k)$, $(i - j + k)$ and $(i + j - k)$, are
(a) $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}}\right)$ (b) $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$ (c) $\left(-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)$ (d) $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ Sol: (d)
Resultant vector $= 2i + 2j + 2k$.
Direction cosines are $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.

7. The position vectors of P and Q are $5i + 4j + ak$ and $-i + 2j - 2k$ respectively. If the distance between them is 7, then the value of a will be
(a) - 5, 1 (b) 5, 1 (c) 0, 5 (d) 1, 0 Sol: (a)
$7 = \sqrt{(5 + 1)^2 + (4 - 2)^2 + (a + 2)^2} \Rightarrow a + 2 = \pm 3$ $a = -5, 1$.

8. A zero vector has
(a) Any direction (b) No direction (c) Many directions (d) None of these Sol: (a)
Direction is not determined.

9. A unit vector a makes an angle $\frac{\pi}{4}$ with z-axis. If $a + i + j$ is a unit vector, then a is equal to

(a) $\frac{\hat{i}}{2}+\frac{\hat{j}}{2}+\frac{\hat{k}}{\sqrt{2}}$ (b) $\frac{\hat{i}}{2}+\frac{\hat{j}}{2}-\frac{\hat{k}}{\sqrt{2}}$ (c) $-\frac{\hat{i}}{2}-\frac{\hat{j}}{2}+\frac{\hat{k}}{\sqrt{2}}$ (d) None of these **Sol: (c)**

Let $\vec{a} = l\hat{i} + m\hat{j} + n\hat{k}$, where $l^2 + m^2 + n^2 = 1$

$\vec{a}$ makes an angle $\frac{\pi}{4}$ with z - axis.

$\therefore n = \frac{1}{\sqrt{2}},\ l^2 + m^2 = \frac{1}{2}$ .....(i)

$\therefore \vec{a} = l\hat{i} + m\hat{j} + \frac{\hat{k}}{\sqrt{2}}$

$\vec{a} + \hat{i} + \hat{j} = (l+1)\hat{i} + (m+1)\hat{j} + \frac{\hat{k}}{\sqrt{2}}$

Its magnitude is 1, hence $(l+1)^2 + (m+1)^2 = \frac{1}{2}$ .....(ii)

$l^2 + 1 + 2l + m^2 + 1 + 2m = \frac{1}{2} \Rightarrow \frac{1}{2} + 2 + 2l + 2m = \frac{1}{2} \Rightarrow l + m = -1 \Rightarrow l = m = \frac{-1}{2}$

From (i) and (ii),

Hence $\vec{a} = -\frac{\hat{i}}{2} - \frac{\hat{j}}{2} + \frac{\hat{k}}{\sqrt{2}}$.

10. A force is a

(a) Unit vector (b) Localized vector (c) Zero vector (d) Free vector **Sol: (b)**

It is a fundamental concept.

ASSERTION AND REASON BASED QUESTIONS

In the following questions, a statement of Assertion (A) is followed by a statement of Reason(R).Pick the correct option:

A) Both Assertion (A) and Reason(R) are true and Reason(R) is the correct explanation of Assertion (A).

B) Both Assertion (A) and Reason(R) are true but Reason(R) is NOT the correct explanation of Assertion (A).

C) Assertion (A) is true but Reason(R) is false.

D) Assertion (A) is false but Reason(R) is true.

1. **Assertion :** $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$ then $\vec{b} = \lambda\vec{a} + \vec{c}$

Reason : If $\vec{a} \times \vec{b} = 0$, $\vec{a}$ is collinear to $\vec{b}$

Sol. **[A]** $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$, $\vec{a} \times (\vec{b} - \vec{c}) = 0$

i.e. $\vec{b} - \vec{c} = \lambda\vec{a} \Rightarrow \vec{b} = \lambda\vec{a} + \vec{c}$

2. **Assertion :** If $\vec{a} \times \vec{b} = \vec{c} \times \vec{d}$, and $\vec{a} \times \vec{c} = \vec{b} \times \vec{d}$ then $\vec{a} - \vec{d}$ is perpendicular to $\vec{b} - \vec{c}$.

Reason : If $\vec{p}$ is perpendicular to $\vec{q}$ then $\vec{p} \cdot \vec{q} = 0$

Sol.[D](Assertion false & reason is true)

$\vec{a} \times \vec{b} = \vec{c} \times \vec{d}$ ... (1)

$\vec{a} \times \vec{c} = \vec{b} \times \vec{d}$ ... (2)

subtract

$\vec{a} \times (\vec{b} - \vec{c}) = (\vec{c} - \vec{b}) \times \vec{d}$; $(\vec{a} - \vec{d}) \times (\vec{b} - \vec{c}) = \vec{0}$; So $\vec{a} - \vec{d}$ is parallel to $\vec{b} - \vec{c}$

3. **Assertion :** If three points P, Q, R have position vectors $\vec{a}, \vec{b}, \vec{c}$ respectively and $2\vec{a} + 3\vec{b} - 5\vec{c} = 0$, then the points P, Q, R must be collinear.

Reason: If for three points A, B, C, ; $\overrightarrow{AB} = \lambda\,\overrightarrow{AC}$, then the points A, B, C must be collinear.

Sol. [A] $2\vec{a}+3\vec{b}-5\vec{c}=0$, $3(\vec{b}-\vec{a})=5(\vec{c}-\vec{a})$

$\vec{b}-\vec{a}=5/3(\vec{c}-\vec{a})$, $\overrightarrow{AB}=5/3\ \overrightarrow{AC}$

$\overrightarrow{AB}$ & $\overrightarrow{AC}$ must be parallel since there is common point A. The points A, B , C must be collinear.

4. **Assertion:** If the difference of two unit vectors is again a unit vector then angle between them is 60^0

Reason : If angle between $\vec{a}$ & $\vec{b}$ is acute than $|\vec{a}.\vec{b}|<|\vec{a}|\,|\vec{b}|$

Sol. [B]

5. **Assertion :** $\vec{a}=\hat{i}+p\hat{j}+2\hat{k}$ and $\vec{b}=2\hat{i}+3\hat{j}+q\hat{k}$ are parallel vector. If p =3/2, q = 4.

Reason: If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ are parallel $\frac{a_1}{b_1}=\frac{a_2}{b_2}=\frac{a_3}{b_3}$

Sol. [A]

6. **Assertion:** If $\vec{a}$ & $\vec{b}$ are unit vectors & θ is the angle between them, then $\sin\frac{\theta}{2}=\frac{|\vec{a}-\vec{b}|}{2}$

Reason : The number of vectors of unit length perpendicular to the vectors $\vec{a}=\hat{i}+\hat{j}$ & $\vec{b}=\hat{j}+\hat{k}$ is two :

Sol.[B] Assertion: $|\vec{a}-\vec{b}|^2=|\vec{a}|^2+|\vec{b}|^2-2|\vec{a}||\vec{b}|\cos\theta=2(1-\cos\theta)$ $(|\vec{a}|=|\vec{b}|=1)$

$|\vec{a}-\vec{b}|^2=2.2\sin^2\frac{\theta}{2}\Rightarrow\sin\frac{\theta}{2}=\frac{|\vec{a}-\vec{b}|}{2}$

Reason: Number of vectors of unit length perpendicular to the vectors $\vec{a}$ & $\vec{b}$ are two,

i.e.$\pm\frac{(\vec{a}\times\vec{b})}{|\vec{a}\times\vec{b}|}$

7. **Assertion:** If $\bar{a}\times\bar{b}=\bar{0}$ and $\bar{a}.\bar{b}=\bar{0}$ then either $\bar{a}=\bar{0}$ or $\bar{b}=\bar{0}$

Reason: $\bar{a}\times\bar{b}=\bar{0}$, $\bar{a}=\bar{0}$, $\bar{b}=\bar{0}$ or $\bar{a}$ is parallel to $\bar{b}$.

$\bar{a}.\bar{b}=0$, $\bar{a}=\bar{0}$, $\bar{b}=\bar{0}$ or $\bar{a}$ is perpendicular to $\bar{b}$.

Sol: A

Since $\bar{a}$ cannot be both parallel and perpendicular to $\bar{b}$, we have $\bar{a}=0$ or $\bar{b}=0$.

8. **Assertion:** If $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ then equation $\vec{r}\times(2\hat{i}-\hat{j}+3\hat{k})=3\hat{i}+\hat{k}$ represent a straight line.

Reason: If $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$, then equation $\vec{r}\times(\hat{i}+2\hat{j}-3\hat{k})=2\hat{i}-\hat{j}$ represent a straight line

Sol.[D] Reason:

$$\vec{r}\times(\hat{i}+2\hat{j}-3\hat{k})=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ x & y & z\\ 1 & 2 & -3\end{vmatrix}=\hat{i}(-3y-2z)-\hat{j}(-3x-z)+\hat{k}(2x-y)$$

$-3y-2z=2,\ 3x+z=-1,\ 2x-y=0$

i.e. $-6x+2z=2,\ 3x+z=-1$

Straight line $2x-y=0,\ 3x+z=-1$

Assertion :

$$\vec{r}\times(2\hat{i}-\hat{j}+3\hat{k})=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ x & y & z\\ 2 & -1 & 3\end{vmatrix}=\hat{i}(3y+z)-\hat{j}(3x-2z)+\hat{k}(-x-2y)$$

[illegible]

[illegible] are parallel planes

[illegible] is not a straight line.

9. Assertion: Angle between $\hat{i}$ and $(\hat{i}+\hat{j})$ is 45°

Reason: $\hat{i}+\hat{j}$ is equally inclined to both $\hat{i}$ and $\hat{j}$ and the angle between $\hat{i}$ and $\hat{j}$ is 90°

so $\theta = \cos^{-1}\frac{1}{\sqrt{2}}$

10. Assertion: If $\vec{a} = \hat{i}+2\hat{j}+3\hat{k}$ and $\vec{b} = -\hat{i}-2\hat{j}-3\hat{k}$, then $|\vec{a}| = |\vec{b}|$

Reason: If $|\vec{a}| = |\vec{b}|$, then it necessarily implies that $\vec{a} = \pm\vec{b}$. Sol. (C)

$|\vec{a}| = |\vec{b}|$ need not imply $\vec{a} = \pm\vec{b}$ eg. $\vec{a} = \hat{i}+2\hat{j}+3\hat{k}$ and $\vec{b} = 2\hat{i}+\hat{j}+3\hat{k}$

11. Assertion: For two non-zero vectors $\vec{a}$ and $\vec{b}$, we have $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$

Reason: For two non-zero vectors $\vec{a}$ and $\vec{b}$, we have $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$

Sol. (B) Fundamental concept.

VERY SHORT ANSWER TYPE QUESTION

1. Given $\vec{AB} = 3\hat{i} - \hat{j} - 5\hat{k}$ and co-ordinates of the terminal point B (0, 1, 3). Find the coordinates of the initial point.

SOL. Given $\vec{AB} = 3\hat{i} - \hat{j} - 5\hat{k}$

$B(0,1,3) \Rightarrow \vec{OB} = \hat{j} + 3\hat{k}$ then $\vec{OA} = ?$

We know that,

$\vec{AB} = \vec{OB} - \vec{OA} \quad \vec{OA} = \vec{OB} - \vec{AB}$

$\vec{OA} = \hat{j} + 3\hat{k} - (3\hat{i} - \hat{j} - 5\hat{k}) = \hat{j} + 3\hat{k} - 3\hat{i} + \hat{j} + 5\hat{k} = -3\hat{i} + 2\hat{j} + 8\hat{k}$

2. If $|\vec{a}| = 2$, $|\vec{b}| = 5$ and $|\vec{a}\times\vec{b}| = 8$, find the value of $\vec{a}\cdot\vec{b}$.

SOL. Given $|\vec{a}| = 2$, $|\vec{b}| = 5$ and $|\vec{a}\times\vec{b}| = 8$, $\vec{a}\cdot\vec{b} = ?$

$|\vec{a}||\vec{b}|\sin\theta = 8 \Rightarrow (2)(5)\sin\theta = 8$

Then, $\sin\theta = \frac{4}{5}$ and $\cos\theta = \frac{3}{5}$

$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta = (2)(5)\frac{3}{5} = 6$

3. If $\vec{a}, \vec{b}$ are any two unit vectors and θ is the angle between them, then show that $\sin\left(\frac{\theta}{2}\right) = \frac{1}{2}|\vec{a} - \vec{b}|$

SOL. Given $|\vec{a}| = 1$, $|\vec{b}| = 1$. $\quad \vec{a}\cdot\vec{a} = |\vec{a}||\vec{a}|\cos 0^\circ = |\vec{a}|^2 = \vec{a}^2$

$|\vec{a} - \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a}\cdot\vec{b} = (1)^2 + (1)^2 - 2|\vec{a}||\vec{b}|\cos\theta$

$= 1 + 1 - 2(1)(1)\cos\theta = 2(1 - \cos\theta)$

$= 2\left(2\sin^2\frac{\theta}{2}\right) \Rightarrow |\vec{a} - \vec{b}| = 2\sin\frac{\theta}{2} \quad , \sin\left(\frac{\theta}{2}\right) = \frac{1}{2}|\vec{a} - \vec{b}|$

4. If $\vec{a} = \hat{i} + 2\hat{j} - \hat{k}$ and $\vec{b} = 3\hat{i} + \hat{j} - 5\hat{k}$, find a unit vector in the direction of $\vec{a} - \vec{b}$

SOL. Given: $\vec{a} = \hat{i} + 2\hat{j} - \hat{k}$ and $\vec{b} = 3\hat{i} + \hat{j} - 5\hat{k}$

Then, $\quad \vec{a} - \vec{b} = \hat{i} + 2\hat{j} - \hat{k} - (3\hat{i} + \hat{j} - 5\hat{k}) = -2\hat{i} + \hat{j} + 4\hat{k} = \vec{c}$ (say)

A unit vector in the direction of $\vec{c}$ is $\frac{\vec{c}}{|\vec{c}|}$

Unit vector in the direction of $\vec{c} = \frac{\vec{c}}{|\vec{c}|} = \frac{-2\hat{i} + \hat{j} + 4\hat{k}}{\sqrt{4+1+16}} = \frac{-2}{\sqrt{21}}\hat{i} + \frac{1}{\sqrt{21}}\hat{j} + \frac{4}{\sqrt{21}}\hat{k}$

5. **If the position vectors of the points A and B are $2\hat{i} + 3\hat{j} - \hat{k}$ and $3\hat{i} + 2\hat{j} + \hat{k}$ then find the vector of magnitude 6 units in the direction of $\overline{AB}$**

SOL: Given $\overline{OA} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\overline{OB} = 3\hat{i} + 2\hat{j} + \hat{k}$ then $\overline{AB} = \overline{OB} - \overline{OA} = \hat{i} - \hat{j} + 2\hat{k}$

A Unit vector in the direction of $\overrightarrow{AB} = \frac{\overrightarrow{AB}}{|\overrightarrow{AB}|} = \frac{\hat{i} - \hat{j} + 2\hat{k}}{\sqrt{1+1+4}} = \frac{\hat{i}}{\sqrt{6}} - \frac{\hat{j}}{\sqrt{6}} + \frac{2\hat{k}}{\sqrt{6}}$

The vector of magnitude 6 units in the direction of $\overrightarrow{AB} = 6\left(\frac{\hat{i}}{\sqrt{6}} - \frac{\hat{j}}{\sqrt{6}} + \frac{2\hat{k}}{\sqrt{6}}\right)$

6. **If P (1, 5, 4) and Q (4, 1, -2), find the direction ratios of $\overrightarrow{PQ}$**

SOL: Let P (1, 5, 4) and Q (4, 1, -2), then $\overrightarrow{OP} = \hat{i} + 5\hat{j} + 4\hat{k}$ $\quad \overrightarrow{OQ} = 4\hat{i} + \hat{j} - 2\hat{k}$

and $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = 3\hat{i} - 4\hat{j} - 6\hat{k}$.

The direction ratios of $\overrightarrow{PQ} = 3, -4, -6$

7. **Find λ, if the vectors $3\hat{i} + \hat{j} - 5\hat{k}$ and $3\hat{i} + \hat{j} - \lambda\hat{k}$ are parallel**

SOL: The vectors $3\hat{i} + \hat{j} - 5\hat{k}$ and $3\hat{i} + \hat{j} - \lambda\hat{k}$ are parallel

Then, $\frac{3}{3} = \frac{1}{1} = \frac{-5}{-\lambda}$ $\qquad$ So, $\lambda = 5$

8. **Find λ, if the vectors $3\hat{i} - \hat{j} - 5\hat{k}$ and $2\hat{i} + 3\hat{j} - \lambda\hat{k}$ are perpendicular**

SOL: Let $\bar{a} = 3\hat{i} - \hat{j} - 5\hat{k}$ and $\bar{b} = 2\hat{i} + 3\hat{j} - \lambda\hat{k}$

If $\bar{a}$ and $\bar{b}$ are perpendicular then $\bar{a} \cdot \bar{b} = 0$

$\rightarrow 3(2) - 1(3) - 5(-\lambda) = 0 \Rightarrow 5\lambda = -3$ then $\lambda = \frac{-3}{5}$

9. **If $\bar{a} = 4\hat{i} + 2\hat{j} - \hat{k}$ and $\vec{b} = 5\hat{i} + 2\hat{j} - 3\hat{k}$ find the angle between $\bar{a} + \vec{b}$ and $\bar{a} - \vec{b}$**

SOL: Let $\vec{a} = 4\hat{i} + 2\hat{j} - \hat{k}$ and $\vec{b} = 5\hat{i} + 2\hat{j} - 3\hat{k}$

Then $\bar{a} + \vec{b} = 9\hat{i} + 4\hat{j} - 4\hat{k} = \vec{c}$, $\qquad \bar{a} - \vec{b} = -\hat{i} + 2\hat{k} = \vec{d}$

So, the angle between $\vec{c}$ and $\vec{d}$ = $\cos\theta = \frac{\vec{c}.\vec{d}}{|\vec{c}||\vec{d}|} = \frac{9(-1) + 4(0) - 4(2)}{\sqrt{81+16+16}\sqrt{1+4}} = \frac{(-17)}{\sqrt{113}\sqrt{5}}$

10. **Find the projection of $\bar{a} = \hat{i} + \hat{j} + 4\hat{k}$ on $\vec{b} = 2\hat{i} + \hat{j} + 2\hat{k}$**

SOL: Given the projection of $\bar{a} = \hat{i} + \hat{j} + 4\hat{k}$ on $\vec{b} = 2\hat{i} + \hat{j} + 2\hat{k}$

Projection of $\bar{a}$ on $\vec{b} = \frac{\bar{a}.\vec{b}}{|\vec{b}|} = \frac{1(2) + 1(1) + 4(2)}{\sqrt{4+1+4}} = \frac{11}{3}$

SHORT ANSWER TYPE QUESTIONS

1. **Find the position vector of a point R which divided the line segment joining the points P and Q with position vectors $\hat{i} + 2\hat{j} - \hat{k}$ and $-\hat{i} + \hat{j} + \hat{k}$ respectively in the ration 2 : 1 i) internally ii) externally.**

SOL: The position vector of a point R which divided the line segment joining the points P and Q with position vectors $\hat{i} + 2\hat{j} - \hat{k}$ and $-\hat{i} + \hat{j} + \hat{k}$ respectively in the ration 2 : 1

Position Vector of a Point R is $\overline{OR} = \frac{m\vec{b} + n\vec{a}}{m+n} = \frac{2(-\hat{i}+\hat{j}+\hat{k}) + 1(\hat{i}+2\hat{j}-\hat{k})}{2+1} = \frac{-2\hat{i}+2\hat{j}+2\hat{k}+\hat{i}+2\hat{j}-\hat{k}}{3} = \frac{-\hat{i}+4\hat{j}+\hat{k}}{3}$

2. **Show that the points A (2, 6, 3), B (1, 2, 7) and C (3, 10, -1) are collinear.**

SOL: Given points A (2, 6, 3), B (1, 2, 7) and C (3, 10, −1)

Then, $\overline{OA} = 2\hat{i} + 6\hat{j} + 3\hat{k}$, $\quad \overline{OB} = \hat{i} + 2\hat{j} + 7\hat{k}$, $\quad \overline{OC} = 3\hat{i} + 10\hat{j} - \hat{k}$

$\overline{AB} = \overline{OB} - \overline{OA} = -\hat{i} - 4\hat{j} + 4\hat{k}$ $\quad \overline{AC} = \overline{OC} - \overline{OA} = \hat{i} + 4\hat{j} - 4\hat{k}$

$\overline{AB} = (-2)\overline{AC}$, $\overline{AB}$ is parallel to $\overline{AC}$ and A is common point. (After D.Rs of two vectors are in proportional) So, the given points A, B, C are collinear

3. If $\bar{a}$, $\bar{b}$ and $\bar{c}$ are three - unit vectors such that $\bar{a} + \bar{b} + \bar{c} = \bar{0}$, prove that $\bar{a} \times \bar{b} = \bar{b} \times \bar{c} = \bar{c} \times \bar{a}$.

SOL: Given $\bar{a}$, $\bar{b}$ and $\bar{c}$ are three vectors such that $\bar{a} + \bar{b} + \bar{c} = 0$ $\bar{a} + \bar{b} = -\bar{c}$

By pre cross multiplication of $\bar{a}$ and $\bar{b}$ on both the sides respectively, we have

$\bar{a}\times(\bar{a}+\bar{b}) = -\bar{a}\times\bar{c}$ implies $\bar{a}\times\bar{a}+\bar{a}\times\bar{b} = -\bar{a}\times\bar{c}$ that is $\bar{a}\times\bar{b} = \bar{c}\times\bar{a}$ ....(1)

$\bar{b}\times(\bar{a}+\bar{b}) = -\bar{b}\times\bar{c} \Rightarrow$ $\bar{b}\times\bar{a}+\bar{b}\times\bar{b} = -\bar{b}\times\bar{c}$ $\Rightarrow \bar{a}\times\bar{b} = \bar{b}\times\bar{c}$ ....(2)

From (1) and (2), we conclude that $\bar{a}\times\bar{b} = \bar{b}\times\bar{c} = \bar{c}\times\bar{a}$

4. **Find the area of the parallelogram with diagonals $3\hat{i} + \hat{j} + 2\hat{k}$ and $\hat{i} - 3\hat{j} + 4\hat{k}$.**

SOL: If $\overline{d_1}$ and $\overline{d_2}$ are diagonals of Parallelogram then Area $= \frac{1}{2}\left|\overline{d_1}\times\overline{d_2}\right|$

$$\overline{d_1}\times\overline{d_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & 2 \\ 1 & -3 & 4 \end{vmatrix}$$

$= \hat{i}(4+6) - \hat{j}(12-2) + \hat{k}(-9-1) = 10\hat{i} - 10\hat{j} - 10\hat{k}$

$\left|\overline{d_1}\times\overline{d_2}\right| = \sqrt{100+100+100} = 10\sqrt{3}$

Area of Parallelogram $= \frac{1}{2}\left|\overline{d_1}\times\overline{d_2}\right| = 5\sqrt{3}$ square units

5. **Find the unit vector perpendicular to vector $\bar{a} = \hat{i} - 7\hat{j} + 7\hat{k}$ and $\bar{b} = 3\hat{i} - 2\hat{j} + 2\hat{k}$**

SOL: Given $\bar{a} = \hat{i} - 7\hat{j} + 7\hat{k}$, $\bar{b} = 3\hat{i} - 2\hat{j} + 2\hat{k}$

A Unit vector perpendicular to $\bar{a}$ and $\bar{b}$ $= \hat{n} = \dfrac{\bar{a}\times\bar{b}}{|\bar{a}\times\bar{b}|}$

$$\bar{a}\times\bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -7 & 7 \\ 3 & -2 & 2 \end{vmatrix}$$

$= \hat{i}(-14+14) - \hat{j}(2-21) + \hat{k}(-2+21) = 19\hat{j} - 19\hat{k}$

$|\bar{a}\times\bar{b}| = \sqrt{(19)^2 + (19)^2} = 19\sqrt{2}$

A Unit vector perpendicular to $\bar{a}$ and $\bar{b}$ $= \hat{n} = \dfrac{\bar{a}\times\bar{b}}{|\bar{a}\times\bar{b}|} = \dfrac{19\hat{j}-19\hat{k}}{19\sqrt{2}} = \dfrac{\hat{j}-\hat{k}}{\sqrt{2}}$

6. **Show that $(\bar{a}-\bar{b})\times(\bar{a}+\bar{b}) = 2(\bar{a}\times\bar{b})$**

SOL: LHS $= (\bar{a}-\bar{b})\times(\bar{a}+\bar{b})$ Note: $\bar{a}\times\bar{b} = -\bar{b}\times\bar{a}$

$= (\bar{a}\times\bar{a}) + (\bar{a}\times\bar{b}) - (\bar{b}\times\bar{a}) - (\bar{b}\times\bar{b}) = 0 + (\bar{a}\times\bar{b}) + (\bar{a}\times\bar{b}) - 0$

$= 2(\bar{a}\times\bar{b})$

7. **If $\bar{a}$, $\bar{b}$ are any two unit vectors and θ is the angle between them, then show that $\cos\left(\frac{\theta}{2}\right) = \frac{1}{2}|\bar{a}+\bar{b}|$**

SOL: Given $|\bar{a}| = 1$, $|\bar{b}| = 1$, we have $\bar{a}.\bar{a} = |\bar{a}||\bar{a}|\cos 0° = |\bar{a}|^2 = a^2$

$|\bar{a}+\bar{b}|^2 = |\bar{a}|^2 + |\bar{b}|^2 + 2\bar{a}\cdot\bar{b}$

$= (1)^2 + (1)^2 + 2|\bar{a}||\bar{b}|\cos\theta$; $= 1 + 1 + 2(1)(1)\cos\theta$

$= 2(1+\cos\theta)$ $= 2\left(2\cos^2\frac{\theta}{2}\right)$

$|\bar{a}+\bar{b}| = 2\cos\frac{\theta}{2}$ then $\cos\left(\frac{\theta}{2}\right) = \frac{1}{2}|\bar{a}+\bar{b}|$

5. **If the position vectors of the points A and B are $2\hat{i}+3\hat{j}-\hat{k}$ and $3\hat{i}+2\hat{j}+\hat{k}$ then find the vector of magnitude 6 units in the direction of $\overline{AB}$**

SOL: Given $\overrightarrow{OA}=2\hat{i}+3\hat{j}-\hat{k}$ and $\overrightarrow{OB}=3\hat{i}+2\hat{j}+\hat{k}$ then $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\hat{i}-\hat{j}+2\hat{k}$

A Unit vector in the direction of $\overrightarrow{AB}=\frac{\overrightarrow{AB}}{|\overrightarrow{AB}|}=\frac{\hat{i}-\hat{j}+2\hat{k}}{\sqrt{1+1+4}}=\frac{\hat{i}}{\sqrt{6}}-\frac{\hat{j}}{\sqrt{6}}+\frac{2\hat{k}}{\sqrt{6}}$

The vector of magnitude 6 units in the direction of $\overrightarrow{AB}=6\left(\frac{\hat{i}}{\sqrt{6}}-\frac{\hat{j}}{\sqrt{6}}+\frac{2\hat{k}}{\sqrt{6}}\right)$

6. **If P (1, 5, 4) and Q (4, 1, -2), find the direction ratios of $\overrightarrow{PQ}$**

SOL: Let P (1, 5, 4) and Q (4, 1, -2), then $\overrightarrow{OP}=\hat{i}+5\hat{j}+4\hat{k}$ $\overrightarrow{OQ}=4\hat{i}+\hat{j}-2\hat{k}$

and $\overrightarrow{PQ}=\overrightarrow{OQ}-\overrightarrow{OP}=3\hat{i}-4\hat{j}-6\hat{k}$.

The direction ratios of $\overrightarrow{PQ}=3,-4,-6$

7. **Find λ, if the vectors $3\hat{i}+\hat{j}-5\hat{k}$ and $3\hat{i}+\hat{j}-\lambda\hat{k}$ are parallel**

SOL: The vectors $3\hat{i}+\hat{j}-5\hat{k}$ and $3\hat{i}+\hat{j}-\lambda\hat{k}$ are parallel

Then, $\frac{3}{3}=\frac{1}{1}=\frac{-5}{-\lambda}$ So, $\lambda=5$

8. **Find λ, if the vectors $3\hat{i}-\hat{j}-5\hat{k}$ and $2\hat{i}+3\hat{j}-\lambda\hat{k}$ are perpendicular**

SOL: Let $\vec{a}=3\hat{i}-\hat{j}-5\hat{k}$ and $\vec{b}=2\hat{i}+3\hat{j}-\lambda\hat{k}$

If $\vec{a}$ and $\vec{b}$ are perpendicular then $\vec{a}\bullet\vec{b}=0$

$\rightarrow 3(2)-1(3)-5(-\lambda)=0 \Rightarrow 5\lambda=-3$ then $\lambda=\frac{-3}{5}$

9. **If $\vec{a}=4\hat{i}+2\hat{j}-\hat{k}$ and $\vec{b}=5\hat{i}+2\hat{j}-3\hat{k}$ find the angle between $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$**

SOL: Let $\vec{a}=4\hat{i}+2\hat{j}-\hat{k}$ and $\vec{b}=5\hat{i}+2\hat{j}-3\hat{k}$

Then $\vec{a}+\vec{b}=9\hat{i}+4\hat{j}-4\hat{k}=\vec{c}$, $\vec{a}-\vec{b}=-\hat{i}+2\hat{k}=\vec{d}$

So, the angle between $\vec{c}$ and $\vec{d}$ = $\cos\theta=\frac{\vec{c}.\vec{d}}{|\vec{c}||\vec{d}|}=\frac{9(-1)+4(0)-4(2)}{\sqrt{81+16+16}\sqrt{1+4}}=\frac{(-17)}{\sqrt{113}\sqrt{5}}$

10. **Find the projection of $\vec{a}=\hat{i}+\hat{j}+4\hat{k}$ on $\vec{b}=2\hat{i}+\hat{j}+2\hat{k}$**

SOL: Given the projection of $\vec{a}=\hat{i}+\hat{j}+4\hat{k}$ on $\vec{b}=2\hat{i}+\hat{j}+2\hat{k}$

Projection of $\vec{a}$ on $\vec{b}=\frac{\vec{a}.\vec{b}}{|\vec{b}|}=\frac{1(2)+1(1)+4(2)}{\sqrt{4+1+4}}=\frac{11}{3}$

SHORT ANSWER TYPE QUESTIONS

1. **Find the position vector of a point R which divided the line segment joining the points P and Q with position vectors $\hat{i}+2\hat{j}-\hat{k}$ and $-\hat{i}+\hat{j}+\hat{k}$ respectively in the ration 2 : 1 i) internally ii) externally.**

SOL: The position vector of a point R which divided the line segment joining the points P and Q with position vectors $\hat{i}+2\hat{j}-\hat{k}$ and $-\hat{i}+\hat{j}+\hat{k}$ respectively in the ration 2 : 1

Position Vector of a Point R is $\overrightarrow{OR}=\frac{m\vec{b}+n\vec{a}}{m+n}=\frac{2(-\hat{i}+\hat{j}+\hat{k})+1(\hat{i}+2\hat{j}-\hat{k})}{2+1}=\frac{-2\hat{i}+2\hat{j}+2\hat{k}+\hat{i}+2\hat{j}-\hat{k}}{3}=\frac{-\hat{i}+4\hat{j}+\hat{k}}{3}$

2. **Show that the points A (2, 6, 3), B (1, 2, 7) and C (3, 10, -1) are collinear.**

SOL: Given points A (2, 6, 3), B (1, 2, 7) and C (3, 10, –1)

Then, $\overrightarrow{OA}=2\hat{i}+6\hat{j}+3\hat{k}$, $\overrightarrow{OB}=\hat{i}+2\hat{j}+7\hat{k}$, $\overrightarrow{OC}=3\hat{i}+10\hat{j}-\hat{k}$

$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=-\hat{i}-4\hat{j}+4\hat{k}$ $\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=\hat{i}+4\hat{j}-4\hat{k}$

$\overline{AB} = (-1)\,\overline{AC}$, $\overline{AB}$ is parallel to $\overline{AC}$ and A is common point. (Alter D.Rs of two vectors are in proportional) So, the given points A, B, C are collinear

3. **If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three - unit vectors such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, prove that $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$.**

SOL: Given $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three vectors such that $\vec{a} + \vec{b} + \vec{c} = 0$ $\vec{a} + \vec{b} = -\vec{c}$

By pre cross multiplication of $\vec{a}$ and b on both the sides respectively, we have

$\vec{a} X (\vec{a} + \vec{b}) = -\vec{a} X \vec{c}$ implies $\vec{a} X \vec{a} + \vec{a} X \vec{b} = -\vec{a} X \vec{c}$ that is $\vec{a} X \vec{b} = \vec{c} X \vec{a}$ ....(1)

$\vec{b} X (\vec{a} + \vec{b}) = -\vec{b} X \vec{c} \Rightarrow$ $\vec{b} X \vec{a} + \vec{b} X \vec{b} = -\vec{b} X \vec{c}$ $\Rightarrow \vec{a} X \vec{b} = \vec{b} X \vec{c}$ ....(2)

From (1) and (2), we conclude that $\vec{a} X \vec{b} = \vec{b} X \vec{c} = \vec{c} X \vec{a}$

4. **Find the area of the parallelogram with diagonals $3\hat{i} + \hat{j} + 2\hat{k}$ and $\hat{i} - 3\hat{j} + 4\hat{k}$.**

SOL: If $\overline{d_1}$ and $\overline{d_2}$ are diagonals of Parallelogram then Area $= \frac{1}{2} |\overline{d_1} X \overline{d_2}|$

$$\overline{d_1} X \overline{d_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & 2 \\ 1 & -3 & 4 \end{vmatrix}$$

$= \hat{i}(4+6) - \hat{j}(12-2) + \hat{k}(-9-1) = 10\hat{i} - 10\hat{j} - 10\hat{k}$

$|\overline{d_1} X \overline{d_2}| = \sqrt{100+100+100} = 10\sqrt{3}$

Area of Parallelogram $= \frac{1}{2} |\overline{d_1} X \overline{d_2}| = 5\sqrt{3}$ square units

5. **Find the unit vector perpendicular to vector $\vec{a} = \hat{i} - 7\hat{j} + 7\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + 2\hat{k}$**

SOL: Given $\vec{a} = \hat{i} - 7\hat{j} + 7\hat{k}$, $\vec{b} = 3\hat{i} - 2\hat{j} + 2\hat{k}$

A Unit vector perpendicular to $\vec{a}$ and $\vec{b}$ = $\hat{n} = \frac{\vec{a} X \vec{b}}{|\vec{a} X \vec{b}|}$

$$\vec{a} X \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -7 & 7 \\ 3 & -2 & 2 \end{vmatrix}$$

$= \hat{i}(-14+14) - \hat{j}(2-21) + \hat{k}(-2+21) = 19\hat{j} - 19\hat{k}$

$|\vec{a} X \vec{b}| = \sqrt{(19)^2 + (19)^2} = 19\sqrt{2}$

A Unit vector perpendicular to $\vec{a}$ and $\vec{b}$ = $\hat{n} = \frac{\vec{a} X \vec{b}}{|\vec{a} X \vec{b}|} = \frac{19\hat{j} - 19\hat{k}}{19\sqrt{2}} = \frac{\hat{j} - \hat{k}}{\sqrt{2}}$

6. **Show that $(\vec{a} - \vec{b}) X (\vec{a} + \vec{b}) = 2(\vec{a} X \vec{b})$**

SOL: LHS $= (\vec{a} - \vec{b}) X (\vec{a} + \vec{b})$ Note: $\vec{a} X \vec{b} = -\vec{b} X \vec{a}$

$= (\vec{a} X \vec{a}) + (\vec{a} X \vec{b}) - (\vec{b} X \vec{a}) - (\vec{b} X \vec{b}) = 0 + (\vec{a} X \vec{b}) + (\vec{a} X \vec{b}) - 0$

$= 2(\vec{a} X \vec{b})$

7. **If $\vec{a}$, $\vec{b}$ are any two unit vectors and θ is the angle between them, then show that $Cos\left(\frac{\theta}{2}\right) = \frac{1}{2}|\vec{a} + \vec{b}|$**

SOL: Given $|\vec{a}| = 1$, $|\vec{b}| = 1$, we have $\vec{a}.\vec{a} = |\vec{a}||\vec{a}| \cos 0° = |\vec{a}|^2 = a^2$

$|\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a} \bullet \vec{b}$

$= (1)^2 + (1)^2 + 2|\vec{a}||\vec{b}| \cos\theta$; $= 1 + 1 + 2(1)(1)\cos\theta$

$= 2(1 + \cos\theta)$ $= 2\left(2\cos^2\frac{\theta}{2}\right)$

$|\vec{a} + \vec{b}| = 2\cos\frac{\theta}{2}$ then $\cos\left(\frac{\theta}{2}\right) = \frac{1}{2}|\vec{a} + \vec{b}|$

8. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are 3 vectors with magnitudes 3,4 and 5 respectively and $\vec{a}+\vec{b}+\vec{c}=\vec{0}$ find $\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}$

SOL: Given $|\vec{a}|=3,\ |b|=5,\ |\vec{c}|=4,\ \vec{a}+\vec{b}+\vec{c}=\vec{0}$

$(\vec{a}+\vec{b}+\vec{c})^2=(\vec{a}+\vec{b}+\vec{c}).(\vec{a}+\vec{b}+\vec{c})$

$=|\vec{a}|^2+|b|^2+|\vec{c}|^2+2\vec{a}\cdot\vec{b}+2\vec{b}\cdot\vec{c}+2\vec{c}\cdot\vec{a}$

$0=9+25+16+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})$

$\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}=\frac{-50}{2}=-25$

9. If $\vec{a}=2\hat{i}+2\hat{j}+3\hat{k}$ $\vec{b}=-\hat{i}+2\hat{j}+\hat{k}$ and $\vec{c}=3\hat{i}+\hat{j}$ are such that $\vec{a}+\lambda\vec{b}$ is perpendicular to $\vec{c}$ then find the value of λ

SOL: Given $\vec{a}=2\hat{i}+2\hat{j}+3\hat{k}$ $\vec{b}=-\hat{i}+2\hat{j}+\hat{k}$ $\vec{c}=3\hat{i}+\hat{j}$

$\vec{a}+\lambda\vec{b}=2\hat{i}+2\hat{j}+3\hat{k}+\lambda(-\hat{i}+2\hat{j}+\hat{k})=(2-\lambda)\hat{i}+(2+2\lambda)\hat{j}+(3+\lambda)\hat{k}$

$\vec{a}+\lambda\vec{b}$ is perpendicular to $\vec{c}\Rightarrow(\vec{a}+\lambda\vec{b}).\vec{c}=0$

$(3)(2-\lambda)+(1)(2+2\lambda)+(0)(3+\lambda)=0\Rightarrow 6-3\lambda+2+2\lambda=0\Rightarrow\lambda=8$

10. If $\vec{a}=\hat{i}+\alpha\hat{j}+2\hat{k}$, $\vec{b}=\beta\hat{i}+\hat{j}-\hat{k}$, are orthogonal and $|\vec{a}|=|\vec{b}|$ then find the values of α and β.

SOL: Given $\vec{a}=\hat{i}+\alpha\hat{j}+2\hat{k}$, $\vec{b}=\beta\hat{i}+\hat{j}-\hat{k}$ are orthogonal and $|\vec{a}|=|\vec{b}|$

$\vec{a}.\vec{b}=0\Rightarrow 1.\beta+\alpha.1+2.-1=0\Rightarrow\beta+\alpha=2$ --- (1) and

$\sqrt{1^2+\alpha^2+2^2}=\sqrt{\beta^2+1^2+(-1)^2}\Rightarrow\alpha^2+5=\beta^2+2\Rightarrow\beta^2-\alpha^2=+3$

Implies $\beta-\alpha=\frac{3}{2}$ --- (2) by solving (1) and (2) $\alpha=\frac{1}{4},\beta=\frac{7}{4}$

CASE BASED TYPE QUESTIONS

1. Relative to a fixed origin O, the points A, B and C have respective position vectors $\hat{i}+10\hat{k}$, $4\hat{i}+3\hat{j}+7\hat{k}$ and $8\hat{i}+7\hat{j}+3\hat{k}$.

(i) Show that A,B and C are collinear, and find the ration AB:BC.

(ii) Calculate the area of the triangle OAC.

Sol:

(i) $\overrightarrow{AB}=(4\hat{i}+3\hat{j}+7\hat{k})-(\hat{i}+10\hat{k})=3\hat{i}+3\hat{j}-3\hat{k}$

$\overrightarrow{BC}=(8\hat{i}+7\hat{j}+3\hat{k})-(4\hat{i}+3\hat{j}+7\hat{k})=4\hat{i}+4\hat{j}-4\hat{k}$

Clearly, the direction ratios of $\overrightarrow{AB}$ and $\overrightarrow{BC}$, B is a common point, so A,B and C are collinear. Ratio of AB:BC is 3:4

(ii) Area of triangle ABC$=\frac{1}{2}|AC||OB|$

$=\frac{1}{2}|(OC)-(OA)|\sqrt{4^2+3^2+7^2}=\frac{1}{2}|(7\hat{i}+7\hat{j}-7\hat{k})|\sqrt{74}$

$=\frac{1}{2}\sqrt{10878}=\frac{7}{2}\sqrt{222}\ sq.units$

2. Let $\vec{a}=2\hat{i}+3\hat{j}-6\hat{k}$, $\vec{b}=2\hat{i}-3\hat{j}+6\hat{k}$ and $\vec{c}=-2\hat{i}+3\hat{j}+6\hat{k}$. Let $\vec{a}_1$ be projection of $\vec{a}$ on $\vec{b}$ and $\vec{a}_2$ be the projection of $\vec{a}_1$ on $\vec{c}$, then

(i) Find the vector $\vec{a}_2$ (ii) Find the value of $\vec{a}_1.\vec{b}$

Sol.

(i) $\vec{a}_1 = \left[(2\hat{i}+3\hat{j}-6\hat{k}).\frac{(2\hat{i}-3\hat{j}+6\hat{k})}{7}\right]\frac{2\hat{i}-3\hat{j}+6\hat{k}}{7} = \frac{-41}{49}(2\hat{i}-3\hat{j}+6\hat{k})$

$\vec{a}_2 = \frac{-41}{49}\left[(2\hat{i}-3\hat{j}+6\hat{k}).\frac{(-2\hat{i}+3\hat{j}+6\hat{k})}{7}\right]\frac{-2\hat{i}+3\hat{j}+6\hat{k}}{7} = \frac{-41}{(49)^2}(-4-9+36)\ (-2\hat{i}+3\hat{j}+6\hat{k})$

$= \frac{943}{49^2}\ (2\hat{i}-3\hat{j}-6\hat{k})$

(ii) $.\vec{a}_1.\vec{b} = \frac{-41}{49}(2\hat{i}-3\hat{j}+6\hat{k})(2\hat{i}-3\hat{j}+6\hat{k}) = -41$.

3. If each of $\vec{a}$, $\vec{b}$, $\vec{c}$ is orthogonal to the sum of the two. Also given $|\vec{a}| = 3, |\vec{b}| = 4, |\vec{c}| = 5$. On the basis of above information, answer the following questions.

(i) If $\vec{a}$ makes angles of equal measures with all three axes, then the tangent of angle becomes.

(ii) If $\vec{a}.\vec{c} = 9$ then the value of $|\vec{a}\times\vec{b}|$

(iii) Find the range of the value $|\vec{a}-\vec{b}|$ (OR)

Find the Value of $|\vec{a}+\vec{b}+\vec{c}|$

Sol: Given that, $|\vec{a}| = 3, |\vec{b}| = 4, |\vec{c}| = 5$-------(1)

And also, $\vec{a}$, $\vec{b}$, $\vec{c}$ is orthogonal to the sum of the two then

$\vec{a}.(\vec{b}+\vec{c}) = 0 \Rightarrow \vec{a}.\vec{b}+\vec{a}.\vec{c} = 0$

$\vec{b}.(\vec{c}+\vec{a}) = 0 \Rightarrow \vec{b}.\vec{c}+\vec{b}.\vec{a} = 0$

$\vec{c}.(\vec{a}+\vec{b}) = 0 \Rightarrow \vec{c}.\vec{a}+\vec{c}.b = 0$

So, by adding above equations then

$2(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}) = 0 ----(2)$

(i) If $\vec{a}$ makes angles of equal measures with all three axes then

$3\cos^2\theta = 1 \Rightarrow \cos\theta = \pm\frac{1}{\sqrt{3}}$ Then $Tan\theta = \pm\sqrt{2}$

(ii) If $\vec{a}.\vec{c} = 9 \Rightarrow 3.4\cos(\vec{a},\vec{c}) = 9 \Rightarrow \cos a = \frac{3}{4}$ $(\vec{a}.\vec{b}+\vec{a}.\vec{c} = 0)$

$\vec{a}.\vec{b} = -\vec{a}.\vec{c} = -9 \Rightarrow |\vec{a}||\vec{b}|\cos(\vec{a},\vec{b}) = -9 \Rightarrow \cos(\vec{a},\vec{b}) = \frac{-3}{4}$

Then, $\sin\left((\vec{a},\vec{b})\right) = \frac{\sqrt{7}}{4}$ $|\vec{a}\times\vec{b}| = 3.4\,sin a = 12.\frac{\sqrt{7}}{4} = 3\sqrt{7}$

then the value of $|\vec{a}\times\vec{b}| = 3\sqrt{7}$

(iii) $|\vec{a}-\vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2 - 2(\vec{a}.\vec{b}) = 9.16 - 2.3.4(\cos\theta) = 25 - 24[-1,1] = [1,49]$

$|\vec{a}-\vec{b}| = [1,7]$ (OR)

$|\vec{a}+\vec{b}+\vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a})$

$= 9+16+25+0 = 50$

Then, $|\vec{a}+\vec{b}+\vec{c}| = \sqrt{50} = 5\sqrt{2}$

CHAPTER-11- THREE DIMENSIONAL GEOMETRY

Gist/Summary of the lesson (Definitions and Formulae)

Condition of perpendicular: If the given lines are perpendicular, then $\theta = 90^\circ$ i.e., $\cos\theta = 0$

$\Rightarrow l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$ or $a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$

Condition of parallelism: If the given lines are parallel, then $\theta = 0^\circ$ i.e., $\sin\theta = 0 \Rightarrow \frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$

Similarly, $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

Intersection of two lines

Determine whether two lines intersect or not. In case they intersect, the following algorithm is used to find their point of intersection.

Algorithm:

Let the two lines be $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ ....(i) and $\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$ ....(ii)

Step I : Write the co-ordinates of general points on (i) and (ii). The co-ordinates of general points on (i) and (ii) are given by $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1} = \lambda$ and $\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2} = \mu$ respectively.

i.e., $(a_1\lambda + x_1, b_1\lambda + y_1, c_1\lambda + z_1)$ and $(a_2\mu + x_2, b_2\mu + y_2, c_2\mu + z_2)$.

Step II : If the lines (i) and (ii) intersect, then they have a common point.

$a_1\lambda + x_1 = a_2\mu + x_2, b_1\lambda + y_1 = b_2\mu + y_2$

and $c_1\lambda + z_1 = c_2\mu + z_2$.

Step III : Solve any two of the equations in λ and μ obtained in step II. If the values of λ and μ satisfy the third equation, then the lines (i) and (ii) intersect, otherwise they do not intersect.

Step IV : To obtain the co-ordinates of the point of intersection, substitute the value of λ (or μ) in the co-ordinates of general point (s) obtained in step I.

Foot of perpendicular from a point $A(\alpha, \beta, \gamma)$ to the line $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ **:** If P be the foot of perpendicular, then P is $(lr + x_1, mr + y_1, nr + z_1)$. Find the direction ratios of AP and apply the condition of perpendicularity of AP and the given line. This will give the value of r and hence the point P, which is foot of perpendicular.

Length and equation of perpendicular : The length of the perpendicular is the distance AP and its equation is the line joining two known points A and P.

The length of the perpendicular is the perpendicular distance of given point from that line.

Reflection or image of a point in a straight line : If the perpendicular PL from point P on the given line be produced to Q such that $PL = QL$, then Q is known as the image or reflection of P in the given line. Also, L is the foot of the perpendicular or the projection of P on the line.

⇒ The number of lines which are equally inclined to the co-ordinate axes is 4.

⇒ If l, m, n are the d.c.'s of a line, then the maximum value of $lmn = \frac{1}{3\sqrt{3}}$.

Distance between two skew lines

(i) $\vec{r_1} = \vec{a}_1 + \lambda\vec{b}_1$ & $\vec{r_2} = \vec{a}_2 + \mu\vec{b}_2$ is $\left|\frac{(\vec{b}_1 \times \vec{b}_2)\cdot(\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|}\right|$ units

(ii) $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ &) $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$ is $\frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}}$

(iii)Parallel lines $\vec{r_1}=\vec{a}_1+\lambda\vec{b}$ & $\vec{r_2}=\vec{a}_2+\mu\vec{b}$ is $\left|\frac{\vec{b}\times(\vec{a_2}-\vec{a}_1)}{|\vec{b}|}\right|$ units

Angle Between two lines

If 'θ' is the acute angle between $\vec{r_1}=\vec{a}_1+\lambda\vec{b_1}$ & $\vec{r_2}=\vec{a}_2+\mu\vec{b}_2$ then,

$\cos\theta=\left|\frac{\vec{b_1}\cdot\vec{b_2}}{|\vec{b_1}||\vec{b_2}|}\right|$.

If $\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$ and $\frac{x-x_1}{l_2}=\frac{y-y_1}{m_2}=\frac{z-z_1}{n_2}$ are the equations of two lines,then acute angle between them is $\cos\theta=|l_1l_2+n_1m_2+n_1n_2|$

Activity to remember the main concepts:

Given form	Standard form of a line in Cartesian form	Standard form of a line in Vector form	Figure	D.r's of a line are	Any point on the line
$\vec{r}=(2+3s)\vec{i}-(s-2)\vec{j}+(1-s)\vec{k}$					
$\frac{2-x}{-3}=\frac{2y-3}{4}=z$					
3x=2y=z					
x=ay+b; z=cy+d					
$\frac{x-2}{1}=\frac{2-y}{-3};z=2$					

MULTIPLE CHOICE TYPE QUESTIONS

1. **If α, β,γ be the angles which a line makes with the positive direction of co-ordinate axes, then** $\sin^2\alpha+\sin^2\beta+\sin^2\gamma=$

   (a) 2 (b) 1 (c) 3 (d) 0

   Sol: a) Since $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$

   $\Rightarrow\Sigma\sin^2\alpha=3-1=2.$

2. **If α,β,γ be the direction angles of a vector and** $\cos\alpha=\frac{14}{15}$, $\cos\beta=\frac{1}{3}$ **then** $\cos\gamma=$

   (a) $\pm\frac{2}{15}$ (b) $\frac{1}{5}$ (c) $\pm\frac{1}{15}$ (d) None of these

   Solution: (a)$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1\Rightarrow\cos\gamma=\sqrt{1-\left(\frac{14}{15}\right)^2-\left(\frac{1}{3}\right)^2}=\sqrt{\frac{8}{9}-\left(\frac{196}{225}\right)}=\pm\frac{2}{15}$.

3. **The direction cosines of the line $x=y=z$ are**

   (a) $\frac{1}{\sqrt3},\frac{1}{\sqrt3},\frac{1}{\sqrt3}$ (b) $\frac{1}{3},\frac{1}{3},\frac{1}{3}$ (c) 1, 1, 1 (d) None of these

   Sol: (a) Direction cosines $\left(\frac{1}{\sqrt3},\frac{1}{\sqrt3},\frac{1}{\sqrt3}\right)$.

4. **If a line makes angles of 30° and 45° with x-axis and y-axis, then the angle made by it with z-axis is**

(a) $45°$ (b) $60°$ (c) $120°$ (d) None of these

Solution: (d) $\cos\gamma = \sqrt{1-\frac{3}{4}-\frac{1}{2}} = \sqrt{\frac{-1}{4}}$, which is not possible.

5. **If the co-ordinates of the points P, Q, R, S be (1, 2, 3), (4, 5, 7), (– 4, 3, – 6) and (2, 0, 2) respectively, then**

(a) $PQ \parallel RS$ (b) $PQ \perp RS$ (c) $PQ = RS$ (d) None of these

Solution: (d)Find angle between the lines PQ and RS, we get that neither $PQ \parallel RS$ nor $PQ \perp RS$. Also $PQ \neq RS$.

6. **If the projections of a line on the co-ordinate axes be 2, –1, 2, then the length of the lines is**

(a) 3 (b) 4 (c) 2 (d) $\frac{1}{2}$

Solution: (a) $r = \sqrt{4+1+4} = 3$.

7. **A line makes angles α, β, γ with the co-ordinate axes. If $\alpha + \beta = 90°$, then $\gamma =$**

(a) 0 (b) $90°$ (c) $180°$ (d) None of these

Solution: (b) Here, $\cos^2\alpha + \cos^2(90-\alpha) + \cos^2\gamma = 1$

$\Rightarrow \cos^2\alpha + \sin^2\alpha + \cos^2\gamma = 1$ $\Rightarrow \cos^2\gamma + 1 = 1 \Rightarrow \gamma = 90°$.

8. **Points (–2, 4, 7), (3, –6, –8) and (1, –2, –2) are**

(a) Collinear (b) Vertices of an equilateral triangle

(c) Vertices of an isosceles triangle (d) None of these

Solution: (a) Here, $\frac{(3-(-2))}{1-3} = \frac{-6-4}{-2-(-6)} = \frac{-8-7}{-2-(-8)}$

$\Rightarrow -\frac{5}{2} = -\frac{5}{2} = -\frac{5}{2}$. Obviously, points are collinear.

9. **The direction ratios of the line joining the points (4, 3, –5) and (–2, 1, –8) are**

(a) $\frac{6}{7}, \frac{2}{7}, \frac{3}{7}$ (b) 6, 2, 3 (c) 2, 4, –13 (d) None of these

Solution: (b) Direction ratios are, $l = 4-(-2) = 6$, $m = 3-1 = 2$ and $n = -5+8 = 3$

a=6, b=2, c=3

10. **The co-ordinates of a point which is equidistant from the points $(0,0,0), (a,0,0), (0,b,0)$ and $(0,0,c)$ are given by**

(a) $\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)$ (b) $\left(-\frac{a}{2}, -\frac{b}{2}, \frac{c}{2}\right)$ (c) $\left(\frac{a}{2}, -\frac{b}{2}, -\frac{c}{2}\right)$ (d) $\left(-\frac{a}{2}, \frac{b}{2}, -\frac{c}{2}\right)$

Solution: (a) Let point be (x, y, z), then $r^2 = x^2 + y^2 + z^2$

$= (x-a)^2 + y^2 + z^2 = x^2 + (y-b)^2 + z^2 = x^2 + y^2 + (z-c)^2$

Therefore $x = \frac{a}{2}, y = \frac{b}{2}$ and $z = \frac{c}{2}$.

11. **The angle between the lines $\frac{x}{1} = \frac{y}{0} = \frac{z}{-1}$ and $\frac{x}{3} = \frac{y}{4} = \frac{z}{5}$ is**

(a) $\cos^{-1}\left(\frac{-1}{5}\right)$ (b) $\cos^{-1}\frac{1}{3}$ (c) $\cos^{-1}\frac{1}{2}$ (d) $\cos^{-1}\frac{1}{4}$

Solution: (a) $\theta = \cos^{-1}\left(\frac{3+0-5}{\sqrt{1+1}\sqrt{9+16+25}}\right) = \cos^{-1}\left(\frac{-2}{10}\right) = \cos^{-1}\left(-\frac{1}{5}\right)$.

12. **If $\frac{x-1}{l} = \frac{y-2}{m} = \frac{z+1}{n}$ is the equation of the line through (1, 2, –1) and (–1, 0, 1), then (l, m, n) is**

(a) (–1, 0, 1) (b) (1, 1, –1) (c) (1, 2, –1) (d) (0, 1, 0)

Solution: (b) $\frac{-2}{l} = \frac{-2}{m} = \frac{2}{n}$; $\therefore$ (l, m, n) are $(1, 1, -1)$.

13. The point of intersection of lines $\frac{x-4}{5}=\frac{y-1}{2}=\frac{z}{1}$ and $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ is

(a) $(-1,-1,-1)$ (b) $(-1,-1,1)$ (c) $(1,-1,-1)$ (d) $(-1,1,-1)$

Solution: (a) Trick : Both lines are satisfied by $(-1,-1,-1)$.

14. The angle between the lines whose direction cosines are proportional to (1, 2, 1) and (2, –3, 6) is

(a) $\cos^{-1}\left(\frac{2}{7\sqrt{6}}\right)$ (b) $\cos^{-1}\left(\frac{1}{7\sqrt{6}}\right)$ (c) $\cos^{-1}\left(\frac{3}{7\sqrt{6}}\right)$ (d) $\cos^{-1}\left(\frac{5}{7\sqrt{6}}\right)$

Solution: (a) $\theta = cos^{-1}\left[\frac{(1)(2)+(2)(-3)+(1)(6)}{\sqrt{1^2+2^2+1^2}\sqrt{2^2+(-3)^2+6^2}}\right]=cos^{-1}\left[\frac{2-6+6}{\sqrt{6}\sqrt{49}}\right] = cos^{-1}\left[\frac{2}{7\sqrt{6}}\right]$.

15. If $P(\bar{p})$, $Q(\bar{q})$, $R(\bar{r})$ and $S(\bar{s})$ be four points such that $3\bar{p}+8\bar{q}=6\bar{r}+5\bar{s}$, then the lines PQ and RS are -

(A) skew (B) intersecting (C) parallel (D) none of these

Sol.[B] Given $3\bar{p}+8\bar{q}=6\bar{r}+5\bar{s}$

$$\frac{3\bar{p}+8\bar{q}}{8+3}=\frac{6\bar{r}+5\bar{s}}{5+6}$$

The point which divides PQ in ratio 8 : 3 is the same as the point which divides RS in the ratio 5 : 6. Hence, the line PQ and RS intersect.

16. The perpendicular distance of the point (2, 4, –1) from the line $\frac{x+5}{1}=\frac{y+3}{4}=\frac{z-6}{-9}$ is

(a) 3 (b) 5 (c) 7 (d) 9

Solution: (c) The perpendicular distance of (2, 4, – 1) from the line $\frac{x+5}{1}=\frac{y+3}{4}=\frac{z-6}{-9}$ is

$$=\left\{(2+5)^2+(4+3)^2+(-1-6)-\left[\frac{1(2+5)+4(4+3)-9(-1-6)}{\sqrt{1+16+81}}\right]^2\right\}^{1/2}$$

$$=\sqrt{147-\left(\frac{98}{\sqrt{98}}\right)^2}=\sqrt{147-98}=\sqrt{49}=7 \text{ (or) } d=\frac{|\overline{AP}\times\bar{b}|}{|\bar{b}|}$$

17. The angle between two lines $\frac{x+1}{2}=\frac{y+3}{2}=\frac{z-4}{-1}$ and $\frac{x-4}{1}=\frac{y+4}{2}=\frac{z+1}{2}$ is (a) $\cos^{-1}\left(\frac{1}{9}\right)$

(b) $\cos^{-1}\left(\frac{2}{9}\right)$ (c) $\cos^{-1}\left(\frac{3}{9}\right)$ (d) $\cos^{-1}\left(\frac{4}{9}\right)$

Solution: (d) $\theta = cos^{-1}\left(\frac{(2)(1)+(2)(2)+(-1)(2)}{\sqrt{2^2+2^2+1^2}\sqrt{1^2+2^2+2^2}}\right) = cos^{-1}\frac{4}{9}$

18. Equation of x-axis is

(a) $\frac{x}{1}=\frac{y}{1}=\frac{z}{1}$ (b) $\frac{x}{0}=\frac{y}{1}=\frac{z}{1}$ (c) $\frac{x}{1}=\frac{y}{0}=\frac{z}{0}$ (d) $\frac{x}{0}=\frac{y}{0}=\frac{z}{1}$

Solution: (c) It is obvious.

19. The angle between the pair of lines with direction ratios (1, 1, 2) and $(\sqrt{3}-1,-\sqrt{3}-1,4)$ is

(a) 30° (b) 45° (c) 60° (d) 90°

Solution: (c) $cos\,\theta = \frac{1(\sqrt{3}-1)-1(\sqrt{3}+1)+2\times 4}{\sqrt{6}\sqrt{24}} = \frac{6}{12} \Rightarrow \theta = 60^o$.

20. The acute angle between the line joining the points (2,1,–3), (–3,1,7) and a line parallel to $\frac{x-1}{3}=\frac{y}{4}=\frac{z+3}{5}$ through the point (–1, 0, 4) is

(a) $\cos^{-1}\left(\frac{7}{5\sqrt{10}}\right)$ (b) $\cos^{-1}\left(\frac{1}{\sqrt{10}}\right)$ (c) $\cos^{-1}\left(\frac{3}{5\sqrt{10}}\right)$ (d) $\cos^{-1}\left(\frac{1}{5\sqrt{10}}\right)$

Solution: (a) Direction ratio of the line joining the point $(2, 1, -3)$, $(-3, 1, 7)$ are (a_1, b_1, c_1)
$\Rightarrow (-3-2, 1-1, 7-(-3)) \Rightarrow (-5, 0, 10)$

Direction ratio of the line parallel to line $\frac{x-1}{3} = \frac{y}{4} = \frac{z+3}{5}$ are $(a_2, b_2, c_2) \Rightarrow (3, 4, 5)$

Angle between two lines,

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}} \qquad \cos\theta = \frac{(-5\times 3)+(0\times 4)+(10\times 5)}{\sqrt{25+0+100}\sqrt{9+16+25}}$$

$$\cos\theta = \frac{35}{25\sqrt{10}} \Rightarrow \theta = \cos^{-1}\left(\frac{7}{5\sqrt{10}}\right).$$

ASSERTION - REASON BASED QUESTIONS

In the following questions, a statement of Assertion (A) is followed by a statement of Reason(R).Pick the correct option:

A) Both Assertion (A) and Reason(R) are true and Reason(R) is the correct explanation of Assertion (A).

B) Both Assertion (A) and Reason(R) are true but Reason(R) is NOT the correct explanation of Assertion (A).

C) Assertion (A) is true but Reason(R) is false.

D) Assertion (A) is false but Reason(R) is true.

1. **ASSERTION:** Equation of a line passes through the point P (2, – 1, 3) & perpendicular to the lines L_1: $\vec{r} = (\hat{i} - \hat{j} + 3\hat{k}) + \alpha(2\hat{i} - 2\hat{j} + \hat{k})$ & L_2: $\vec{r} = (\hat{i} + 3\hat{k}) + \beta(\hat{i} + 2\hat{j} + 2\hat{k})$ both is $\vec{r} = (2\hat{i} - \hat{j} + 3\hat{k}) + s(-6\hat{i} - 2\hat{j} + 6\hat{k})$

REASON: Let, L_1 : $\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ & L_2 : $\vec{r} = \vec{a_2} + \mu\vec{b_2}$. Then equation of a line passing through the point P($\vec{a}$) & perpendicular to the lines L_1 & L_2 both is $\vec{r} = \vec{a} + t(\vec{b_1} \times \vec{b_2})$

Sol.[A] $\vec{a} = 2\hat{i} - \hat{j} + 3\hat{k}$ By L_1 : $\vec{b_1} = 2\hat{i} - 2\hat{j} + \hat{k}$ By L_2 : $\vec{b_2} = \hat{i} + 2\hat{j} + 2\hat{k}$

$$\vec{b_1} \times \vec{b_2} = \begin{vmatrix} i & j & k \\ 2 & -2 & 1 \\ 1 & 2 & 2 \end{vmatrix} = -6\hat{i} - 2\hat{j} + 6\hat{k}$$

Then, required equation of line is $\vec{r} = \vec{a} + t(\vec{b_1} \times \vec{b_2})$

2. **ASSERTION :** If the lines $\vec{r} = \vec{a} + \lambda\vec{b}$ and $\vec{r} = \vec{c} + \mu\vec{d}$ intersects at a point then $(\vec{c} - \vec{a}).(\vec{b} \times \vec{d}) = 0$

REASON: Two coplanar lines always intersects.

Sol. [C]

3. **ASSERTION:** The angle between the rays of with d.r's (4, –3, 5) and (3, 4, 5) is $\frac{\pi}{3}$.

REASON: The angle between the rays whose d.c's are l_1, m_1, n_1 and l_2, m_2, n_2 is given by whose $\cos\alpha = l_1l_2 + m_1m_2 + n_1n_2$

Sol. [B] $\cos\theta = \left(\frac{12-12+25}{\sqrt{50}\sqrt{50}}\right) = \frac{25}{50} = \frac{1}{2} \Rightarrow \theta = \cos^{-1}\frac{1}{2} = \frac{\pi}{3}$

4. **ASSERTION:** A line makes 60° with x-axis and 30° with y-axis then it makes 90° with z-axis.

REASON:: If a ray makes angles α, β, γ with x-axis, y-axis and z-axis respectively then $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 1$

Sol. [C] $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

5. **ASSERTION:** If lines x = ay + b, z = 3y + 4 and x = 2y + 6, z = ay + d are perpendicular to each other then a = -1/5

   REASON: If two lines with d.rs a_1, b_1, c_1 and a_2, b_2, c_2 are perpendicular then $a_1a_2 + b_1b_2 + c_1c_2 = 0$ **Sol. [D]**

   $\frac{x-b}{a} = \frac{y}{1} = \frac{z-4}{3}$ and $\frac{x-6}{2} = \frac{y}{1} = \frac{z-d}{a} \Rightarrow 2a + 1 + 3a = 0 \Rightarrow a = \frac{-1}{5}$

6. **ASSERTION:** Equation of a line passing through the points (1,2,3) and (3,-1,3) is $\frac{x-3}{2} = \frac{y+1}{-3} = \frac{z-3}{0}$.

   REASON: Equation of a line passing through the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is $\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$ .Sol: A

7. **ASSERTION:** A line through the points (4,7,8) and (2,3,4) is parallel to a line through the points (-1,-2,1) and (1,2,5).

   REASON: Lines $\vec{r} = \overline{a_1} + \lambda\overline{b_1}$ and $\vec{r} = \overline{a_2} + \mu\overline{b_2}$ are parallel if $\overline{b_1}.\overline{b_2} = 0$. **Sol: C**

   D.rs are in proportion, hence the lines are parallel but $\overline{b_1}.\overline{b_2} = 0$ gives that lines are perpendicular to each other.

8. **ASSERTION:** Quadrilateral formed by the points A(0,0,0), B(3,4,5), C(8,8,8) and D(5,4,3) is a Rhombus.

   REASON: ABCD is a Rhombus if AB=BC=CD=DA and AC≠BD. **Sol: A**

9. **ASSERTION:** A line in space cannot be drawn perpendicular to x,y and z axes simultaneously.

   REASON: For any line making angles α, β, γ with the positive directions of x,y and z axes respectively, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.Sol: D

10. **Assertion:** The vector equation of the line passing through the points (6, –4, 5) and (3, 4, 1) is $\vec{r} = (6\hat{\imath} - 4\hat{\jmath} + 5\hat{k}) + \mu(-3\hat{\imath} + 8\hat{\jmath} - 4\hat{k})$

    Reason: The vector equation of the line passing through the points $\vec{a}$ and $\vec{b}$ is $\vec{r} = \vec{a} + \mu(\vec{b} - \vec{a})$. Sol: A, It is fundamental concept

VERY SHORT ANSWER TYPE QUESTIONS

1. **Find the points on the line $\frac{x+2}{3} = \frac{y+1}{2} = \frac{z-3}{2}$ at a distance of 5 units from the point P(1,3,3).**

Sol: Given line is $\frac{x+2}{3} = \frac{y+1}{2} = \frac{z-3}{2} = \mu$ then,

General point on the line is $R(3\mu - 2, 2\mu - 1, 2\mu + 3)$

Distance from R to P(1, 3, 3) is 5 units

$$\sqrt{(3\mu-2-1)^2 + (2\mu-1-3)^2 + (2\mu+3-3)^2} = 5$$

$\therefore \mu = 0, 2.$

$\therefore$ R(-2, -1, 3) or R(4 , 3 , 7)

2. **The equation of a line is $5x - 3 = 15y + 7 = 3 - 10z$. Write the direction cosines of the line.**

Sol: Given equation of a line is $5x - 3 = 15y + 7 = 3 - 10z$ ... (1)

Here coefficients of x, y and z are 5, 15 and 10. (without sign)

LCM (5, 15, 10) = 30. Thus, dividing by 30 we have eq. (1) becomes

$$\frac{5x-3}{30}=\frac{15y+7}{30}=\frac{3-10z}{30}$$

$$\frac{5(x-\frac{3}{5})}{30}=\frac{15(y+\frac{7}{15})}{30}=\frac{-10(z-\frac{3}{10})}{30}$$

$$\frac{x-\frac{3}{5}}{6}=\frac{y+\frac{7}{15}}{2}=\frac{z-\frac{3}{10}}{-3} \quad \text{.... (2)}$$

The standard form of equation is given as

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c} \quad \text{.... (3)}$$

Comparing the above standard equation with Eq. (2), we get 6, 2, -3 are the direction ratios of the given line.

Now $\sqrt{6^2+2^2+(-3)^2}=\sqrt{49}=7$

Now, the direction cosines of given line are $\frac{6}{7}, \frac{2}{7}, \frac{-3}{7}$.

3. **If α, β, γ are the angles made by the lines with x,y and z axes then show that $\cos2\alpha+\cos2\beta+\cos2\gamma=-1$**

Sol: If α, β, γ are the angles made by the lines with x,y and z axes then we have $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$

By writing $\cos2x=2\cos^2x-1$

$$\frac{1+\cos2\alpha}{2}+\frac{1+\cos2\beta}{2}+\frac{1+\cos2\gamma}{2}=1 \quad ;\cos2\beta+\cos2\beta+\cos2\gamma+3=2$$

So, $\cos2\alpha+\cos2\beta+\cos2\gamma=-1$

4. **If α, β and γ are the angles which a line makes with positive direction of ,x , y and z axes respectively , then prove that $\sin^2\alpha+\sin^2\beta+\sin^2\gamma=2$.**

Sol : We have $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$

$$\Rightarrow 1-\sin^2\alpha+1-\sin^2\beta+1-\sin^2\gamma=1$$

$$\Rightarrow \sin^2\alpha+\sin^2\beta+\sin^2\gamma=2$$

5. **Find the angle between the pair of lines given by $\frac{x-3}{1}=\frac{y+2}{-2}=\frac{z+4}{2}$; $\frac{x-5}{-3}=\frac{y+2}{2}=\frac{z}{6}$**

Sol: By observing the two lines, the direction ratios of two lines are <1,-2, 2> and <-3,2,6>

Let θ be the angle between the two given lines

Hence, $\cos\theta=\frac{-3-4+12}{3\times7}=\frac{5}{21}\Rightarrow\theta=\cos^{-1}\left(\frac{5}{21}\right)$

6. **If the lines $\frac{x-1}{-3}=\frac{2y-2}{4k}=\frac{3-z}{-2}$ and $\frac{x-1}{3k}=\frac{3y-1}{6}=\frac{z-6}{-5}$ are perpendicular to each other, find the value of k.**

Sol: The equations of the lines in standard form are

$$\frac{x-1}{-3}=\frac{y-1}{2k}=\frac{z-3}{2} \text{ and } \frac{x-1}{3k}=\frac{y-\frac{1}{3}}{2}=\frac{z-6}{-5}$$

So, the direction ratios of two lines are <-3,2k,2> and <3k,2,-5>

Since the lines are perpendicular $\vec{a}.\vec{b}=0$

That is $a_1a_2+b_1b_2+c_1c_2=0$; $-9k+4k-10=0\Rightarrow k=-2$

7. **If the angle between the lines $\frac{x-5}{\alpha}=\frac{y+2}{-5}=\frac{z+\frac{24}{5}}{\beta}$ and $\frac{x}{1}=\frac{y}{0}=\frac{z}{1}$ is $\frac{\pi}{4}$. Find the relation between α and β.**

Sol: The direction ratios of two given lines are < α, -5, β>

and $\theta = \frac{\pi}{4}$ be the angle between the two given lines. Then

$$\cos\frac{\pi}{4} = \frac{\alpha+\beta}{(\sqrt{\alpha^2+\beta^2+25}).\sqrt{2}} \Rightarrow \frac{1}{\sqrt{2}} = \frac{\alpha+\beta}{\sqrt{\alpha^2+\beta^2+25}\ .\sqrt{2}} \Rightarrow \alpha\beta = \frac{25}{2}$$

8. **Write the vector equation of a line passing through point $(1,-1,2)$ and parallel to the line whose equation is $\frac{x-3}{1} = \frac{y-1}{2} = \frac{z+1}{-2}$.**

Sol: The Vector equation of a line passing through a point with position vector $\vec{a}$ and parallel to a given vector $\vec{b}$ is $\vec{r} = \vec{a} + \lambda\vec{b}$, where $\lambda \in R$

Given point on the line is $(1,-1,2)$ then $\vec{a} = \hat{i} - \hat{j} + 2\hat{k}$

and a vector parallel to the line $\frac{(x-3)}{1} = \frac{(y-1)}{2} = \frac{(z+1)}{-2}$ is $\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$

Hence, Direction Ratios of a required line are $1, 2$ and -2

So, required vector equation of line is

$$\vec{r} = (\hat{i} - \hat{j} + 2\hat{k}) + \lambda(\hat{i} + 2\hat{j} - 2\hat{k}), where\ \lambda \in R$$

9. **Find the vector and Cartesian equations of a line that passes through the point $A(1,2,-1)$ and parallel to the line $5x - 25 = 14 - 7y = 35z$.**

Sol: The given line is $\frac{x-5}{\frac{1}{5}} = \frac{y-2}{-\frac{1}{7}} = \frac{z}{\frac{1}{35}}$

hence, Cartesian equation of a line passing through $A(1,2,-1)$ and parallel to given line is

$$\frac{x-1}{\frac{1}{5}} = \frac{y-2}{-\frac{1}{7}} = \frac{z+1}{\frac{1}{35}}\ or\ \frac{x-1}{7} = \frac{y-2}{-5} = \frac{z+1}{1}$$

and the corresponding vector equation is $\vec{r} = \hat{\imath} + 2\hat{\jmath} - \hat{k} + \lambda(7\hat{\imath} - 5\hat{\jmath} + \hat{k})$.

10. **If the equation of a line is $= ay + b, z = cy + d$, then find the direction ratios of the line and a point on the line.**

Sol: The given equation of a line is not in standard form

$x - b = ay\ and\ z - d = cy$ $\quad \frac{x-b}{a} = y\ and\ \frac{z-d}{c} = y$ then $\frac{x-b}{a} = \frac{y-0}{1} = \frac{z-d}{c}$

Hence, the direction ratios are $a, 1, c$ and a point on the line is $(b, 0, d)$

SHORT ANSWER TYPE QUESTIONS

1. **Find the coordinates of the point where the line through the points (–1, 1, –8) and (5, –2, 10) crosses the ZX-plane.**

Sol: The equation of line passing through two given points (–1, 1, –8) and (5, –2, 10) is

$$\frac{x+1}{6} = \frac{y-1}{-3} = \frac{z+8}{18} = \lambda$$

Any point on this line is $(6\lambda - 1, -3\lambda + 1, 18\lambda - 8)$

If a line crosses ZX-plane i.e. co-ordinate of y is zero

$-3\lambda + 1 = 0 \Rightarrow \lambda = \frac{1}{3}$

Substitute value of λ in the point, then Required point on ZX-Plane is $(1, 0, -2)$

2. **Find the shortest distance between the Lines: $\vec{r} = (3\hat{\imath} + 2\hat{\jmath} - 4\hat{k}) + \lambda(\hat{\imath} + 2\hat{\jmath} + 2\hat{k})$ and $\vec{r} = (5\hat{\imath} - \hat{\jmath}) + \mu(3\hat{\imath} + 2\hat{\jmath} + 6\hat{k})$. Also, find whether the lines are intersecting or not.**

Sol: The given two lines are

$\vec{r} = (3\hat{\imath} + 2\hat{\jmath} - 4\hat{k}) + \lambda(\hat{\imath} + 2\hat{\jmath} + 2\hat{k})$ and $\vec{r} = (5\hat{\imath} - \hat{\jmath}) + \mu(3\hat{\imath} + 2\hat{\jmath} + 6\hat{k})$

By observing the given two lines, we have

$\vec{a_1} = 3\hat{\imath} + 2\hat{\jmath} - 4\hat{k}, \vec{b_1} = \hat{\imath} + 2\hat{\jmath} + 2\hat{k}\ ; \vec{a_2} = 5\hat{\imath} - \hat{\jmath}, \vec{b_2} = 3\hat{\imath} + 2\hat{\jmath} + 6\hat{k}$

Hence, Shortest distance between the above lines = $\left|\frac{(\vec{a_2}-\vec{a_1}).(\vec{b_1}\times\vec{b_2})}{|\vec{b_1}\times\vec{b_2}|}\right|$

$$(\vec{a_2}-\vec{a_1}) = 2\hat{\imath} - 3\hat{\jmath} + 4\hat{k}$$

and $\vec{b_1}\times\vec{b_2} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & 2 \\ 3 & 2 & 6 \end{vmatrix} = 8\hat{\imath} + 0\hat{\jmath} - 4\hat{k}$

$|\vec{b_1}\times\vec{b_2}| = \sqrt{80}$ Now, S.D = $\left|\frac{(2\hat{\imath}-3\hat{\jmath}+4\hat{k}).(8\hat{\imath}+0\hat{\jmath}-4\hat{k})}{\sqrt{80}}\right| = \frac{0}{\sqrt{80}} = 0$

If shortest distance is zero then the given two lines are intersecting.

3. Equations of sides of a parallelogram ABCD are as follows

$AB: \frac{x+1}{1} = \frac{y-2}{-2} = \frac{z-1}{2}$ $BC: \frac{x-1}{3} = \frac{y+2}{-5} = \frac{z-5}{3}$ $CD: \frac{x-4}{1} = \frac{y+7}{-2} = \frac{z-8}{2}$

$DA: \frac{x-2}{3} = \frac{y+3}{-5} = \frac{z-4}{3}$.

Find the equation of diagonal BD.

Sol: Let P be any point on line AB is $(\lambda - 1, -2\lambda + 2, 2\lambda + 1)$ and Q be any point on line BC is $(3\mu + 1, -5\mu - 2, 3\mu + 5)$

For some value of λ and μ, both the lines are intersecting at a point B.

$$P = Q$$

$$\Rightarrow (\lambda - 1, -2\lambda + 2, 2\lambda + 1) = (3\mu + 1, -5\mu - 2, 3\mu + 5)$$

B is the point of intersection of AB and BC , so coordinates of B are $(1,-2,5)$

Similarly, any point on line CD is $(\lambda + 4, -2\lambda - 7, 2\lambda + 8)$ and any point on line DA is $(3\mu + 2, -5\mu - 3, 3\mu + 4)$

D is the point of intersection of CD and DA , so coordinates of B are $(2,-3,4)$.

The equation of a line passing through the points $B(1,-2,5)$ *and* $D(2,-3,4)$ *is*

$\frac{x-1}{2-1} = \frac{y-(-2)}{-3-(-2)} = \frac{z-5}{4-5}$; $\frac{x-1}{1} = \frac{y+2}{-1} = \frac{z-5}{-1}$

4. Check whether the line $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-4}{5} = \frac{y-1}{2} = z$ are skew or not.

Sol: From the given two lines, we have

Let $\vec{a_1} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$, $\vec{a_2} = 4\hat{\imath} + \hat{\jmath}$ $\vec{b_1} = 2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}$, $\vec{b_2} = 5\hat{\imath} + 2\hat{\jmath} + \hat{k}$

and $\vec{a_2} - \vec{a_1} = 3\hat{\imath} - \hat{\jmath} - 3\hat{k}$ & $\vec{b_1}\times\vec{b_2} = -5\hat{\imath} + 18\hat{\jmath} - 11\hat{k}$

Now, $(\vec{a_2} - \vec{a_1}).(\vec{b_1}\times\vec{b_2}) = (3\ X - 5) + (-1\ X\ 18) + (-3\ X - 11)$ $= -15 - 18 + 33 = 0$

Hence, given lines are not skew lines.

5. Find the equation of the line which bisects the line segment joining points $A(2,3,4)$ and $B(4,5,8)$ and is perpendicular to the lines $\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7}$ and

$$\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}$$

Sol: The midpoint of line joining points $A(2,3,4)$ and $B(4,5,8)$ is $(3,4,6)$

So, the equation of line passing through $(3,4,6)$ with direction ratios a,b,c is

$\frac{x-3}{a} = \frac{y-4}{b} = \frac{z-6}{c}$

Since this line is perpendicular to $\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7}$ and $\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}$

$\Rightarrow 3a - 16b + 7c = 0 \quad \Rightarrow 3a + 8b - 5c = 0$

by using cross multiplication method, we have a=2,b=3,c=6

Hence, the required equation of a line passing through (3,4,6) with d.r's <2,3,6> is

$$\frac{x-3}{2}=\frac{y-4}{3}=\frac{z-6}{6}$$

6. **Find the vector and Cartesian equations of the line which is perpendicular to the lines with equations $\frac{x+2}{1}=\frac{y-3}{2}=\frac{z+1}{4}$ *and* $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and passes through the point (1, 1, 1).**

Sol: Let equation of line through (1, 1, 1) with direction ratios a,b, c be

$\frac{x-1}{a}=\frac{y-1}{b}=\frac{z-1}{c}$.....................(i)

If Line(i) perpendicular to the lines $\frac{x+2}{1}=\frac{y-3}{2}=\frac{z+1}{4}$ *and* $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$

Therefore, a+2b+4c=0 and 2a+3b+4c=0 $\therefore$ dr's are -4, 4, -1 or 4, -4, 1

$\therefore$ Hence, the Cartesian equation of a line is $\frac{x-1}{4}=\frac{y-1}{-4}=\frac{z-1}{1}$ and vector equation of a line is $r=(i+j+k)+\mu(4i-4j+k)$

7. **Find the value of p for which the lines $\vec{r}=\lambda i+(2\lambda+1)j+(3\lambda+2)\hat{k}$ and $\vec{r}=i-3\mu j+(p\mu+7)\hat{k}$ are perpendicular to each other and also intersect. Also find the point of intersection of the given lines.**

Sol: By writing the given two equations in standard form

The direction ratios of a given two lines are (1,2,3) and (0,-3,p) respectively.

If the lines are perpendicular to each other then $a_1a_2+b_1b_2+c_1c_2=0$

$$\Rightarrow 1\times 0+2\times(-3)+3\times p=0\Rightarrow p=2$$

Any point on the line $\vec{r}=\lambda\hat{i}+(2\lambda+1)\hat{j}+(3\lambda+2)\hat{k}$ is $(\lambda, 2\lambda+1, 3\lambda+2)$

Any point on the line $\vec{r}=\hat{i}-3\mu\hat{j}+(p\mu+7)\hat{k}$ is $(1,-3\mu, 2\mu+7)$

For point of intersection, we can equate the coordinates

$$(\lambda, 2\lambda+1, 3\lambda+2)=(1,-3\mu,2\mu+7)$$

and solving we get $\lambda=1$ and $\mu=-1$

The point of intersection is (1,3,5).

8. **Find the values of p, so that the lines**

$$l_1:\frac{1-x}{3}=\frac{7y-14}{p}=\frac{z-3}{2} \text{ and } l_2:\frac{7-7x}{3p}=\frac{y-5}{1}=\frac{6-z}{5}$$

are perpendicular to each other. Also, find the equation of a line passing through a point (3,2,−4) and parallel to line l_1.

Sol: Writing the given line in standard form as

$$\frac{(x-1)}{-3}=\frac{y-2}{\frac{p}{7}}=\frac{z-3}{2}=r_1(let) \quad ....(1)$$

$$\text{and } \frac{x-1}{\frac{-3p}{7}}=\frac{y-5}{1}=\frac{z-6}{-5}=r_2(let) \quad ....(2)$$

Two lines with DR's a_1,b_1,c_1 and a_2,b_2,c_2 are perpendicular if

$$a_1a_2+b_1b_2+c_1c_2=0$$

Thus line (1) and (2) will intersect at right angle, if

$-3\left(\frac{-3p}{7}\right)+\frac{p}{7}(1)+2(-5)=0$, $\frac{9p}{7}+\frac{p}{7}=10$, $\frac{10p}{7}=10\Rightarrow p=7$

This is the required value of p.

Also, we know that the equation of a line which passes through the point (x_1,y_1,z_1) with direction ratios a, b, c is given by

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$$

Since, required line is parallel to line l_1.

So, $\quad a=-3, b=\frac{7}{7}=1\ and\ c=2$

Now, equation of line passing through the point (3,2,-4) and having direction ratios (-3,1,2) is $\frac{x-3}{-3}=\frac{y-2}{1}=\frac{z+4}{2} \Rightarrow \frac{3-x}{3}=\frac{y-2}{1}=\frac{z+4}{2}$

9. Find the vector and the Cartesian equations of a line passing through the point $(1,2,-4)$ and parallel to the line joining the points $A(3,3,-5)$ and $B(1,0,-11)$. Hence find the distance between the two lines.

Sol: The direction ratios of a line joining the points $A(3,3,-5)$ and $B(1,0,-11)$ are (3-1,3-0, -5-(-11))=(2,3,6)

So, the Vector equation of required line through (1, 2, – 4) with direction ratios (2,3,6) is

$$\vec{r}=\hat{\imath}+2\hat{\jmath}-4\hat{k}+\lambda(2\hat{\imath}+3\hat{\jmath}+6\hat{k})$$

and the cartesian equation is $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z+4}{6}$

Equation of line through A(3, 3, – 5) and B(1, 0, – 11) is

$$\vec{r}=3\hat{\imath}+3\hat{\jmath}-5\hat{k}+\lambda(2\hat{\imath}+3\hat{\jmath}+6\hat{k})$$

Distance between parallel lines is given by $d=\frac{|(\vec{a_2}-\vec{a_1})\times\vec{b}|}{|\vec{b}|}$

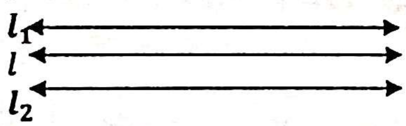
Here $\vec{b}=2\hat{\imath}+3\hat{\jmath}+6\hat{k}$, $\vec{a_1}=\hat{\imath}+2\hat{\jmath}-4\hat{k}$, $\vec{a_2}=3\hat{\imath}+3\hat{\jmath}-5\hat{k}$

$(\vec{a_2}-\vec{a_1})=2\hat{\imath}+\hat{\jmath}-\hat{k}$, $(\vec{a_2}-\vec{a_1})\times\vec{b}=9\hat{\imath}-14\hat{\jmath}+4\hat{k}$

$$|(\vec{a_2}-\vec{a_1})X\vec{b}|=\sqrt{81+196+16}=\sqrt{293}\ and\ |\vec{b}|=\sqrt{4+9+36}=7$$

$$so, d=\frac{\sqrt{293}}{7}.$$

10. Find the equation of a line l_2 which is the mirror image of the line l_1 with respect to line(l) : $\frac{x}{1}=\frac{y-1}{2}=\frac{z-2}{3}$, given that line l_1 passes through the point P(1, 6, 3) and parallel to line l.



Sol : Since l_1 passes through the point P(1,6,3) and parallel to line l equation of l_1 is

$$\frac{x-1}{1}=\frac{y-6}{2}=\frac{z-3}{3}=\mu$$

Since line l_2 is the mirror image of the line l_1 with respect to line l , l_2 is parallel to l.

Foot of perpendicular of P(1,6,3)to line l is (1,3,5)

So point on l_2 is (1,0,7) , image of P(1,6,3) with respect to line l

So, equation of l_2 which passes through (1,0,7) and parallel to l is

$$\frac{x-1}{1}=\frac{y}{2}=\frac{z-7}{3}=\lambda$$

LONG ANSWER TYPE QUESTIONS

1. Find the co-ordinates of the foot of the perpendicular drawn from the point $A(-1,8,4)$ to the line joining points $B(0,-1,3)$ and $C(2,-3,-1)$. Hence find the image of the point A in the line BC.

Sol: The equation of a line joining the points B (0, – 1, 3) and C (2, – 3, – 1) is

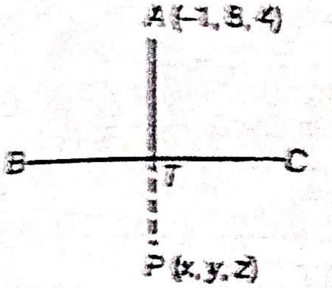
$\vec{r}=(0\hat{\imath}-\hat{\jmath}+3\hat{k})+\lambda[(2\hat{\imath}-3\hat{\jmath}-1\hat{k})-(0\hat{\imath}-\hat{\jmath}+3\hat{k})]$

$\Rightarrow \bar{r} = (-\hat{j} + 3\hat{k}) + \lambda(2\hat{i} - 2\hat{j} - 4\hat{k})$

$\Rightarrow \bar{r} = (2\lambda)\hat{i} + (-2\lambda - 1)\hat{j} + (-4\lambda + 3)\hat{k}, \lambda \in R$

So any point on line BC is to the form $(2\lambda, -2\lambda - 1, -4\lambda + 3)$

Let foot of the perpendicular drawn from point A to the line BC be $T(2\lambda, -2\lambda - 1, -4\lambda + 3)$.



Now, DR's of line AT is $(2\lambda + 1, -2\lambda - 1 - 8, -4\lambda + 3 - 4)$ or $(2\lambda + 1, 2\lambda - 9, -4\lambda - 1)$.

Since, AT is perpendicular to BC.

$\therefore 2 \times (2\lambda + 1) + (-2) \times (-2\lambda - 9) + (-4)(-4\lambda - 1) = 0$

$[\because a_1a_2 + b_1b_2 + c_1c_2 = 0]$

$\Rightarrow 4\lambda + 2 + 4\lambda + 18 + 16\lambda + 4 = 0 \Rightarrow 24\lambda + 24 = 0, \lambda = -1$

$\therefore$ Coordinates of foot of perpendicular is

$T(2 \times (-1)), -2 \times (-1) - 1, -4 \times (-1) + 3)$ or $T(-2, 1, 7)$

Let P(x, y, z) be the image of a point A with respect to the line BC. So, point T is the mid-point of AP.

$\therefore$ Coordinates of T = Coordinates of mid-point of AP

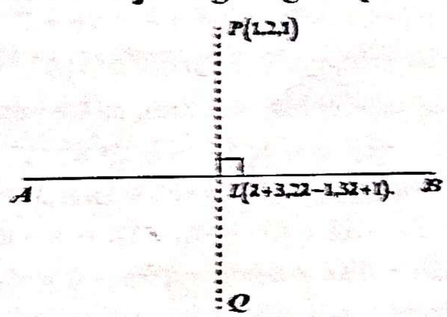
$\Rightarrow (-2, 1, 7) = [(x-1)/2, (y+8)/2, (z+4)/2]$

On equating the corresponding coordinates, we get

$-2 = (x-1)/2$, $1 = (y+8)/2$ and $7 = (z+4)/2$ $\Rightarrow x = -3, y = -6$ and $z = 10$

Hence, Image=(-3,-6,10)

2. Find the image of the point (1,2, 1) with respect to the line $\frac{x-3}{1} = \frac{y+1}{2} = \frac{z-1}{3}$. Also find the equation of the line joining the given point and its image.



Sol: Let P(1,2,1) be the given point and L be the foot of the perpendicular from P to the given line AB (as shown in the figure above).

Let's put $\frac{x-3}{1} = \frac{y+1}{2} = \frac{z-1}{3} = \lambda$. Then, $x = \lambda + 3, y = 2\lambda - 1, z = 3\lambda + 1$

Let the coordinates of the point L be $(\lambda + 3, 2\lambda - 1, 3\lambda + 1)$.

So, direction ratios of PL are$(\lambda + 3 - 1, 2\lambda - 1 - 2, 3\lambda + 1 - 1)$ i.e., $(\lambda + 2, 2\lambda - 3, 3\lambda)$

Direction ratios of the given line are 1,2 and 3, which is perpendicular to PL. Therefore, we have,

$$(\lambda+2)\cdot 1+(2\lambda-3)\cdot 2+3\lambda\cdot 3=0\Rightarrow 14\lambda=4\Rightarrow \lambda=2/7$$

Then, $\lambda+3=\frac{2}{7}+3=\frac{23}{7}$; $2\lambda-1=2\left(\frac{2}{7}\right)-1=-\frac{3}{7}$; $3\lambda+1=3\left(\frac{2}{7}\right)+1=\frac{13}{7}$

Therefore, coordinates of the point L are $\left(\frac{23}{7},-\frac{3}{7},\frac{13}{7}\right)$.

Let $Q(x_1,y_1,z_1)$ be the image of P(1,2,1) with respect to the given line. Then, L is the mid-point of PQ.

Therefore, $\frac{1+x_1}{2}=\frac{23}{7},\frac{2+y_1}{2}=-\frac{3}{7},\frac{1+z_1}{2}=\frac{13}{7}\Rightarrow x_1=\frac{39}{7},y_1=-\frac{20}{7},z_1=\frac{19}{7}$

Hence, the image of the point P(1,2,1) with respect to the given line Q $\left(\frac{39}{7},-\frac{20}{7},\frac{19}{7}\right)$.

The equation of the line joining P(1,2,1) and Q $\left(\frac{39}{7},-\frac{20}{7},\frac{19}{7}\right)$ is

$$\frac{x-1}{32/7}=\frac{y-2}{-34/7}=\frac{z-1}{12/7}\Rightarrow\frac{x-1}{16}=\frac{y-2}{-17}=\frac{z-1}{6}$$

3. Find the shortest distance between the lines l_1 and l_2 whose vector equations are $\vec{r}=(-\hat{i}-\hat{j}-\hat{k})+\lambda(7\hat{i}-6\hat{j}+\hat{k})$ and $\vec{r}=(3\hat{i}+5\hat{j}+7\hat{k})+\mu(\hat{i}-2\hat{j}+\hat{k})$. where λ and μ are parameters.

Sol: Given that equation of lines are

$$\vec{r}=(-\hat{i}-\hat{j}-\hat{k})+\lambda(7\hat{i}-6\hat{j}+\hat{k})\ ...\ (i)\ and$$
$$\vec{r}=(3\hat{i}+5\hat{j}+7\hat{k})+\mu(\hat{i}-2\hat{j}+\hat{k})\ ......\ (ii)$$

The given lines are non-parallel lines as vectors $7\hat{i}-6\hat{j}+\hat{k}$ and $\hat{i}-2\hat{j}+\hat{k}$ are not parallel. There is a unique line segment PQ (lying on line (i) and Q on the other line (ii), which is at right angles to both the lines PQ&is the shortest distance between the lines.

Hence, the shortest possible distance between the lines $=PQ$.

Let the position vector of the point P lying on the line $\vec{r}=(-\hat{i}-\hat{j}-\hat{k})+\lambda(7\hat{i}-6\hat{j}+\hat{k})$ where' λ' is a scalar, is $(7\lambda-1)\hat{i}-(6\lambda+1)\hat{j}+(\lambda-1)\hat{k}$, for some λ and the position vector of the point Q lying on the line $\vec{r}=(3\hat{i}+5\hat{j}+7\hat{k})+\mu(\hat{i}-2\hat{j}+\hat{k})$ where ' μ ' is a scalar, is $(\mu+3)\hat{i}+(-2\mu+5)\hat{j}+(\mu+7)\hat{k}$, for some μ. Now, the vector

$$\overrightarrow{PQ}=\overrightarrow{OQ}-\overrightarrow{OP}$$
$$\overrightarrow{PQ}=(\mu+3-7\lambda+1)\hat{i}+(-2\mu+5+6\lambda+1)\hat{j}+(\mu+7-\lambda+1)k$$

i.e., $\overrightarrow{(PQ)}=(\mu-7\lambda+4)\hat{i}+(-2\mu+6\lambda+6)\hat{j}+(\mu-\lambda+8)\hat{k}$;

(where ' O ' is the origin), is perpendicular to both the lines, so the vector $\overrightarrow{PQ}$ is perpendicular to both the vectors $7\hat{i}-6\hat{j}+\hat{k}$ and $\hat{i}-2\hat{j}+\hat{k}$.

$$\Rightarrow(\mu-7\lambda+4)\cdot 7+(-2\mu+6\lambda+6)\cdot(-6)+(\mu-\lambda+8)\cdot 1=0$$
$$\&(\mu-7\lambda+4)\cdot 1+(-2\mu+6\lambda+6)\cdot(-2)+(\mu-\lambda+8)\cdot 1=0$$
$$\Rightarrow 20\mu-86\lambda=0 \Rightarrow 10\mu-43\lambda=0\&6\mu-20\lambda=0\Rightarrow 3\mu-10\lambda=0$$

On solving the above equations, we get $\mu=\lambda=0$

So, the position vector of the points P and Q are $-\hat{i}-\hat{j}-\hat{k}$ and $3\hat{i}+5\hat{j}+7\hat{k}$ respectively.

$\overrightarrow{(PQ)}=4\hat{i}+6\hat{j}+8\hat{k}$ and $|\overrightarrow{(PQ)}|=\sqrt{(4^2+6^2+8^2)}=\sqrt{116}=2\sqrt{29}$ units.

(OR) Find the shortest distance between the lines

$\frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1}$ and $\frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1}$

SOL: Given lines: $\frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1}$ and $\frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1}$

Shortest Distance between the lines $= \frac{|(\vec{a_2}-\vec{a_1}).(\vec{b_1} \times \vec{b_2})|}{|\vec{b_1} \times \vec{b_2}|}$

$\vec{a_1} = -\hat{\imath} - \hat{\jmath} - \hat{k} \quad \vec{a_2} = 3\hat{\imath} + 5\hat{\jmath} + 7\hat{k} \quad \vec{a_2} - \vec{a_1} = 4\hat{\imath} + 6\hat{\jmath} + 8\hat{k}$

$\vec{b_1} = 7\hat{\imath} - 6\hat{\jmath} + \hat{k} \quad \vec{b_2} = \hat{\imath} - 2\hat{\jmath} + \hat{k}$

$$\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 7 & -6 & 1 \\ 1 & -2 & 1 \end{vmatrix} = \hat{\imath}(-6+2) - \hat{\jmath}(7-1) + \hat{k}(-14+6) = -4\hat{\imath} - 6\hat{\jmath} - 8\hat{k}$$

$|\vec{b_1} \times \vec{b_2}| = \sqrt{16+36+64} = \sqrt{116}$

$$SD = \frac{|(\vec{a_2}-\vec{a_1}).(\vec{b_1} \times \vec{b_2})|}{|\vec{b_1} \times \vec{b_2}|} = \left|\frac{(4\hat{\imath}+6\hat{\jmath}+8\hat{k}).(-4\hat{\imath}-6\hat{\jmath}-8\hat{k})}{\sqrt{116}}\right|$$

$$SD = \left|\frac{(4)(-4)+(6)(-6)+(8)(-8)}{\sqrt{116}}\right| = \left|\frac{-16-36-64}{\sqrt{116}}\right| = \frac{116}{\sqrt{116}} = \sqrt{116} = 2\sqrt{29}$$

4. Find the shortest distance between the lines whose vector equations are

$\vec{r} = (1-t)\hat{\imath} + (2-t)\hat{\jmath} + (3-2t)\hat{k}$ & $\vec{r} = (s+1)\hat{\imath} + (2s-1)\hat{\jmath} - (2s+1)\hat{k}$

SOL: The equations of the given lines are

$\vec{r} = (1-t)\hat{\imath} + (2-t)\hat{\jmath} + (3-2t)\hat{k}$ & $\vec{r} = (s+1)\hat{\imath} + (2s-1)\hat{\jmath} - (2s+1)\hat{k}$

After writing standard equation of a line, then we have

$\vec{r} = \hat{\imath} - 2\hat{\jmath} + 3\hat{k} + t(-\hat{\imath} + \hat{\jmath} - 2\hat{k})$ & $\vec{r} = \hat{\imath} - \hat{\jmath} - \hat{k} + s(\hat{\imath} + 2\hat{\jmath} - 2\hat{k})$

Shortest Distance between the lines $= \frac{|(\vec{a_2}-\vec{a_1}).(\vec{b_1} \times \vec{b_2})|}{|\vec{b_1} \times \vec{b_2}|}$

$\vec{a_1} = \hat{\imath} - 2\hat{\jmath} + 3\hat{k} \quad \vec{a_2} = \hat{\imath} - \hat{\jmath} - \hat{k} \quad \vec{a_2} - \vec{a_1} = 0\hat{\imath} + \hat{\jmath} - 4\hat{k}$

$\vec{b_1} = -\hat{\imath} + \hat{\jmath} - 2\hat{k} \quad \vec{b_2} = \hat{\imath} + 2\hat{\jmath} - 2\hat{k}$

$$\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & 1 & -2 \\ 1 & 2 & -2 \end{vmatrix} = \hat{\imath}(-2+4) - \hat{\jmath}(2+2) + \hat{k}(-2-1) = 2\hat{\imath} - 4\hat{\jmath} - 3\hat{k}$$

$|\vec{b_1} \times \vec{b_2}| = \sqrt{4+16+9} = \sqrt{29}$

$$SD = \frac{|(\vec{a_2}-\vec{a_1}).(\vec{b_1} \times \vec{b_2})|}{|\vec{b_1} \times \vec{b_2}|} = \left|\frac{(0\hat{\imath}+\hat{\jmath}-4\hat{k}).(2\hat{\imath}-4\hat{\jmath}-3\hat{k})}{\sqrt{29}}\right|$$

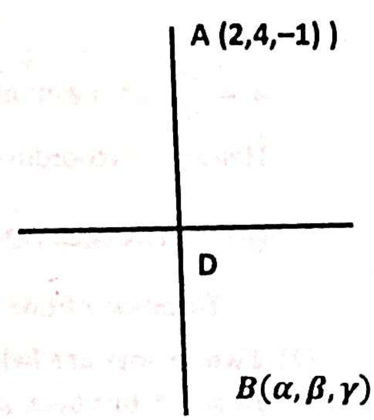
$$SD = \left|\frac{(0)(2)+(1)(-4)+(-4)(-3)}{\sqrt{29}}\right| = \left|\frac{-4+12}{\sqrt{29}}\right| = \frac{8}{\sqrt{29}} = \frac{8\sqrt{29}}{29}$$

5). Find the image of the point (2, 4, –1) in the line $\frac{x+5}{1} = \frac{y+3}{4} = \frac{z-6}{-9}$

SOL: Given point (2, 4, –1)

Given line $\frac{x+5}{1} = \frac{y+3}{4} = \frac{z-6}{-9}$

Let $\frac{x+5}{1} = \frac{y+3}{4} = \frac{z-6}{-9} = k$



Any Point on given line D (k–5, 4k–3, –9k+6)

DRs of AD: $x_2 - x_1, y_2 - y_1, z_2 - z_1$

k–5 –2, 4k–3 –4, –9k+6 +1

k–7, 4k–7, –9k+7

DRs of the given line $\frac{x+5}{1} = \frac{y+3}{4} = \frac{z-6}{-9}$ are 1, 4, –9

AD is perpendicular to given line: $a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$

$1(k-7) + 4(4k-7) + (-9)(-9k+7)$

$k - 7 + 16k - 28 + 81k - 63 = 0$

$98k - 98 = 0$

$k = 1$

Substitute k=1 in D then
Foot of the perpendicular = D (−4, 1, −3)
Let B(α, β, γ) be the image of A
Then mid-point of AB = D

$\left(\frac{\alpha+2}{2}, \frac{\beta+4}{2}, \frac{\gamma-1}{2}\right) = (-4, 1, -3)$

$\frac{\alpha+2}{2} = -4,\quad \alpha+2 = -8,\quad \alpha = -10$

$\frac{\beta+4}{2} = 1\quad \beta+4 = 2\quad \beta = -2$

$\frac{\gamma-1}{2} = -3\quad \gamma-1 = -6\quad \gamma = -5$

Image = B (−10, −2, −5)

CASE STUDY BASED QUESTIONS

1) Imagine you are at a point A, a café you visit often. Your friend is at the point B, a bookstall a few blocks away on a straight road. You want to meet your friend at a point on the line joining the café and the bookstall. Another friend, who is at home on the other side of the same road represented by point P, also wants to join. You decide to determine the exact meeting point by finding the foot of the perpendicular from P on the line joining the café and the bookstall. Given that co-ordinates of the café (A) are (1,2,4), of the bookstall (B) are (3,4,5) and the home are (2,1,3).

(i) Find the location of the meeting point.

(ii) Find the distance from Home to meeting point? Also, find the equation of the path connected by café and bookstall.

Sol: Let Q be the foot of the perpendicular from P(2,1,3) to AB and let Q divide AB in the ratio k:1.

Then, co-ordinates of Q are $\left(\frac{3k+1}{k+1}, \frac{4k+2}{k+1}, \frac{5k+4}{k+1}\right)$.

Direction ratios of PQ are $\frac{3k+1}{k+1} - 2, \frac{4k+2}{k+1} - 1, \frac{5k+4}{k+1} - 3$

$= \frac{k-1}{k+1}, \frac{3k+1}{k+1}, \frac{2k+1}{k+1}$

Direction ratios of AB are <3-1, 4-2, 5-4>; =<2, 2, 1>

As, $PQ \perp AB$,

$$2\left(\frac{k-1}{k+1}\right) + 2\left(\frac{3k+1}{k+1}\right) + 1\left(\frac{2k+1}{k+1}\right) = 0$$

$k = \frac{-1}{10}$ and substitute value of k in point Q.

Hence, the co-ordinates of Q are $\left(\frac{7}{9}, \frac{16}{9}, \frac{35}{9}\right)$.

(ii) Distance PQ= $\sqrt{\left(\frac{7}{9} - 2\right)^2 + \left(\frac{16}{9} - 1\right)^2 + \left(\frac{35}{9} - 3\right)^2} = \sqrt{\frac{121}{81} + \frac{49}{81} + \frac{64}{81}} = \frac{\sqrt{234}}{9}$ *units*

Equation of line AB is $\frac{x-1}{2} = \frac{y-2}{2} = \frac{z-4}{1}$

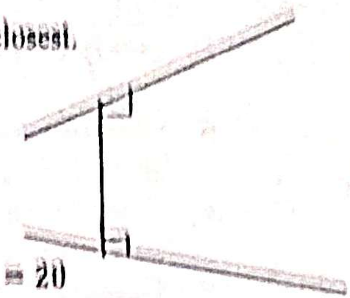
(2) Two drones are being used for radar centre analysis over an area of enemy land. Drone A has been programmed to fly on the path given by $\vec{r} = 6\hat{i} + 2\hat{j} + 2\hat{k} + \mu(\hat{i} - 2\hat{j} + 2\hat{k})$ and drone B has been programmed to fly on the path $\vec{r} = -4\hat{i} - \hat{k} + \beta(\widehat{3i} - 2\hat{j} - 2\hat{k})$.

(i) At what points on their respective paths should they reach, so that they will be closest to each other?

(ii) **Find the shortest distance between these paths.**

Sol: Let P and Q be the points on their respective paths when they are closest.

∴ *PQ is ⊥ to both line 1 and line 2*



So, general point on line-1 is $P=(\lambda+6,-2\lambda+2,\lambda+2)$

General point on line-2 is $Q=(3\mu-4,-2\mu,-2\mu-1)$

Direction ratios of PQ are $<3\mu-\lambda-10,-2\mu+2\lambda-2,-2\mu-2\lambda-3>$

Line -1 is perpendicular to PQ, then

$\Rightarrow -3\lambda+\mu=4$ *and line − 2 is perpendicular to PQ then* $-3\lambda+17\mu=20$

By solving the above equations, we have $\mu=1$ *and* $\lambda=-1$

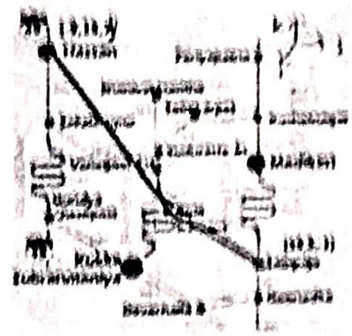
Hence, the points are P=(5,4,0) and Q=(-1,-2,-3).

(iii) $PQ=\sqrt{(5+1)^2+(4+2)^2+(0+3)^2}=\sqrt{36+36+9}=\sqrt{81}=9$ *units*

3) Two tunnels are planned to be dug through

4) Bisle ghat to improve traffic infrastructure.

5) (Shown in the figure-not to Scale).



Digging at one end of the tunnel is to begin at the point (−9,15,7) at Hassan and continue in the direction 7i-5j-k. The digging at the other end of the tunnel will start at the coordinate (33, 5, −1) near Sampaje and continue in the direction −14i +3k. Both sections are to be straight lines. The coordinates are measured relative to a fixed origin O, where one unit is 500 meters.

(i) Show that the two sections of the tunnel will eventually meet at a point near Bisle Ghat and find the coordinates of this point.

(ii) Find the total length of the two tunnels to the nearest kilometre. (OR)

Find the coordinates of the point which is nearest to the origin on the line

$\vec{r}=(\vec{i}+2\vec{j}+2\vec{k})+\mu(-\vec{i}-3\vec{j}+0\vec{k})$

Sol:

i) Let the equation the lines (paths) are

Hassan to Bisle:

$l_1:\frac{x+9}{7}=\frac{y-15}{-5}=\frac{z-7}{-1}=\lambda$

Bisle to Sampaje $l_2:\frac{x-33}{-14}=\frac{y-5}{0}=\frac{z1}{3}=\mu$

Any points on the line 1 and 2 are respectively

$P=(7\lambda-9,-5\lambda+15,-\lambda+7)$ $Q=(-14\mu+33,5,3\mu-1)$

For some values of λ and , lines are intersecting, so P=Q

$7\lambda-9=-14\mu+33,-5\lambda+15=5,-\lambda+7=3\mu-1$ Hence, $\mu=2,\lambda=2$

So, 5=5 hence the lines are intersecting and point of intersection is (5,5,5)

ii) (Let $\overline{OA}=-9\hat{\imath}+15\hat{\jmath}+7\hat{k}$ *and* $\overline{OB}=33\hat{\imath}+5\hat{\jmath}-\hat{k}$

Then, $\overrightarrow{AB}=42\hat{\imath}-10\hat{\jmath}-8\hat{k}$

$|\overrightarrow{AB}|=\sqrt{1764+100+64}=\sqrt{1928}\cong 44\ units=44X500\ mtrs=22000\ mtrs$

(OR)

The coordinates of any point on the line are $P=(1-\mu,2-3\mu,2))$

Distance from (0,0,0) to the point P is

$OP=\sqrt{(1-\mu)^2+(2-3\mu)^2+(2)^2}=\sqrt{1+\mu^2-2\mu+4+9\mu^2-12\mu+4}$

$D=\sqrt{10\mu^2-14\mu+9}$, Let $f=10\mu^2-14\mu+9$

Hence, minimum value occurs at $\mu=\frac{-(-14)}{2(10)}=\frac{7}{10}$

Substitute value of μ *in P then* $P=(\frac{3}{10},\frac{-1}{10},2)$